

BURKHOLDER'S MARTINGALE TRANSFORM

PAATA IVANISHVILI

ABSTRACT. We find the sharp constant $C = C(\tau, p, \mathbb{E}G/\mathbb{E}F)$ of the following inequality $\|(G^2 + \tau^2 F^2)^{1/2}\|_p \leq C\|G\|_p$, where G is the transform of a martingale F under a predictable sequence ε with absolute value 1, $1 < p < 2$, and τ is any real number.

CONTENTS

1. Introduction	2
1.1. Our main results	2
1.2. Plan of the paper	4
2. Definitions and the setting of the problem	4
3. Homogeneous Monge–Ampère equation and minimal concave functions	8
3.1. Foliation	8
3.2. Cup	16
4. Construction of the Bellman function	23
4.1. Reduction to the two dimensional case	23
4.2. Construction of a candidate for M	25
4.3. Concavity in another direction	30
5. Sharp constants via foliation	34
5.1. Main theorem	34
5.2. Case $y_p \leq s_0$.	36
5.3. Case $y_p > s_0$.	37
6. Extremizers via foliation	40
6.1. Case $s_0 \leq y_p$.	41
6.2. Case $s_0 > y_p$.	44
Acknowledgements	48
References	48

2010 *Mathematics Subject Classification.* 42B20, 42B35, 47A30.

Key words and phrases. Martingale transform, Bellman function, Monge–Ampère equation, Concave envelopes, Developable surface, Torsion.

1. INTRODUCTION

In this paper we find the sharp estimates for perturbation of martingale transform. Let I be an interval of the real line \mathbb{R} , and let $|I|$ be its Lebesgue length. By symbol \mathcal{B} we denote the Borel sigma algebra on this interval. Let $\{F_n\}_{n=0}^\infty$ be a martingale on the probability space $(I, \mathcal{B}, dx/|I|)$ with a filtration $I = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \dots \subset \mathcal{F}$. Consider any sequence of functions $\{\varepsilon_n\}_{n=1}^\infty$ such that ε_n is \mathcal{F}_{n-1} measurable and $|\varepsilon_n| \leq 1$. Set

$$G_n \stackrel{\text{def}}{=} G_0 + \sum_{k=1}^n \varepsilon_k (F_k - F_{k-1}) \quad \forall n \geq 1.$$

$\{G_n\}_{n=0}^\infty$ is called martingale transform of $\{F_n\}$, where $G_0 = \text{const}$ on I . Surely $\{G_n\}_{n=0}^\infty$ is a martingale with the same filtration $\{\mathcal{F}_n\}_{n=0}^\infty$.

In [8] Burkholder proved that for if $|G_0| \leq |F_0|$, for any p , $1 < p < \infty$, we have

$$(1) \quad \|G_n\|_{L^p} \leq (p^* - 1) \|F_n\|_{L^p} \quad \forall n \geq 0,$$

where $p^* = \max\{p - 1, \frac{1}{p-1}\}$, and $p^* - 1$ in (1) is sharp. Burkholder also showed that it is sufficient to prove inequality (1) for the sequences of numbers $\{\varepsilon_n\}$ such that $\varepsilon_n = \pm 1$, $\forall n \geq 1$. It was also mentioned that such estimate as (1) does not depend on the choice of filtration $\{\mathcal{F}_n\}$, for example, one can consider only dyadic filtration. For more information of estimate (1) we refer the reader to [8], [9].

In [5] a bit more general estimate was obtained by Bellman function technique and Monge–Ampère equation, namely, estimate (1) holds if and only if

$$(2) \quad |G_0| \leq (p^* - 1) |F_0|.$$

Further we assume that $\{\varepsilon_n\}$ is a predictable sequence of functions such that $|\varepsilon_n| = 1$.

In [4], a perturbation of the martingale transform was investigated. Namely, under the same assumptions as (2) it was proved that for $2 \leq p < \infty$, $\tau \in \mathbb{R}$, we have

$$(3) \quad \|(G_n^2 + \tau^2 F_n^2)^{1/2}\|_{L^p} \leq ((p^* - 1)^2 + \tau^2)^{1/2} \|F_n\|_{L^p}, \quad \forall n \geq 0,$$

where the constant $((p^* - 1)^2 + \tau^2)^{1/2}$ is sharp. It was also announced as proven that the same sharp estimate holds for $1 < p < 2$, $|\tau| \leq 0.5$ and the case $1 < p < 2$, $|\tau| > 0.5$ was left open. For a motivation of a study of a perturbed martingale transform we refer the reader to [4]. We should mention that Burkholder's method [8] and the Bellman function technique approach [5], [4] have similar traces in the sense that both of them reduce the required estimate to finding a certain minimal diagonally concave function with prescribed boundary conditions. However, the methods of construction of such function are different, unlike Burkholder's method [8], in [5], [4] construction of the function is based on the Monge–Ampère equation.

1.1. Our main results. Firstly, we should mention that the proof of (3) presented in [4] is not correct in the case $1 < p < 2$, $0 < |\tau| \leq 0.5$ (the constructed function does not satisfy necessary concavity condition).

In the present paper we obtain the sharp estimate of the perturbed martingale transform for the the remaining case $1 < p < 2$ and for all $\tau \in \mathbb{R}$, moreover, we do not require the condition $|G_0| \leq (p^* - 1)|F_0|$.

Set

$$u(z) \stackrel{\text{def}}{=} \tau^p(p-1)(\tau^2 + z^2)^{(2-p)/2} - \tau^2(p-1) + (1+z)^{2-p} - z(2-p) - 1.$$

Theorem 1. *Let $1 < p < 2$, and let $\{G_n\}_{n=0}^\infty$ be a martingale transform of $\{F_n\}_{n=0}^\infty$. Set $\beta' = \frac{|G_0| - |F_0|}{|G_0| + |F_0|}$. The following estimates are sharp*

1. *If $u\left(\frac{1}{p-1}\right) \leq 0$ then*

$$\|(\tau^2 F_n^2 + G_n^2)^{1/2}\|_{L^p} \leq \left(\tau^2 + \max \left\{ \left| \frac{G_0}{F_0} \right|, \frac{1}{p-1} \right\}^2 \right)^{\frac{1}{2}} \|F_n\|_{L^p}, \quad \forall n \geq 0.$$

2. *If $u\left(\frac{1}{p-1}\right) > 0$ then*

$$\|(\tau^2 F_n^2 + G_n^2)^{1/2}\|_{L^p}^p \leq C(\beta') \|F_n\|_{L^p}^p, \quad \forall n \geq 0,$$

where $C(\beta')$ is continuous nondecreasing, and it is defined by the following way:

$$C(\beta') \stackrel{\text{def}}{=} \begin{cases} \left(\tau^2 + \frac{|G_0|^2}{|F_0|^2} \right)^{p/2}, & \beta' \geq s_0; \\ \frac{\tau^p}{1 - \frac{2^{2-p}(1-s_0)^{p-1}}{(\tau^2+1)(p-1)(1-s_0)+2(2-p)}}, & \beta' \leq -1 + \frac{2}{p}; \\ C(\beta') & \beta' \in (-1 + 2/p, s_0); \end{cases}$$

where $s_0 \in (-1 + 2/p, 1)$ is the solution of the equation $u\left(\frac{1+s_0}{1-s_0}\right) = 0$.

Explicit expression for the function $C(\beta')$ on the interval $(-1 + 2/p, s_0)$ was hard to express in a simple way. The reader can find the value of the function $C(\beta')$ in Theorem 2, part (ii).

Remark 1. *The condition $u\left(\frac{1}{p-1}\right) \leq 0$ is always true for example when $|\tau| \leq 0.822$. So we also obtain Burkholder's result in the limit case when $\tau = 0$. It is worth mentioning that although the proof of estimate (3) is wrong in [4], the announced result in the case $1 < p < 2$, $|\tau| < 0.5$ remains true by virtue of Theorem 1.*

It is worth mentioning that one of the important result of the current paper is that we find the function (6), and the above estimates are corollaries of this result. We also would like to mention that unlike [5], [4] style of writing current article is different. Instead of doing a lot of technical computations and checking which case is valid, we present some pure geometrical facts regarding minimal concave functions with prescribed boundary conditions, and by this way we avoid such type of computations, moreover we explain to the reader how do we construct our Bellman function (6) based on these geometrical facts derived in Section 3.

1.2. Plan of the paper. In Section 2 our aim is to explain how to reduce estimate of the type (3) to the finding of a certain function with required properties. *The answer* to this question is well-known and can be found in [4]. Slightly different function was investigated in [5], however it possess almost the same properties and the proof works exactly in the same way. Although these answers are known, nevertheless, these arguments are repeated here for completeness.

The reader familiar with the Bellman function technique or with the estimates of the martingale transform can skip Section 2, we only mention that we look for the minimal diagonally concave function $H(x_1, x_2, x_3)$ (see Definition 3) in the domain $\Omega = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : |x_1|^p \leq x_3\}$ with a boundary condition $H(x_1, x_2, |x_1|^p) = (x_2^2 + \tau^2 x_1^2)^{p/2}$, and instantly begin with section 3.

Section 3 is devoted to the investigation of the minimal concave functions of two variables. It is worth mentioning that the first crucial steps in this direction for some special cases were made in [6]. In Section 3 we develop this theory for slightly more general case, we investigate some special foliation called the *cup* and another useful object, called *force functions*.

We should note that the theory of minimal concave functions of two variables does not include the minimal diagonally concave functions of three variables, nevertheless, this knowledge allows us to construct the candidate for H in Section 4, but with some additional technical work not mentioned in Section 3.

In section 5 we find the good estimates for the perturbed martingale transform. In Section 6 we prove that a candidate for H constructed in Section 4 coincides with H , and as a corollary we show sharpness of the estimates found for perturbed martingale transform in Section 5.

In conclusion, the reader can note that the hard technical part of the current paper lies in the construction of the minimal diagonally concave function of three variables with the given boundary condition. We also should mention that this procedure answers on the following questions: why there exists at least one nontrivial diagonally concave function with the given boundary condition, how to choose and construct the minimal one among these functions and why our Bellman function is actually minimal among all such continuous functions.

2. DEFINITIONS AND THE SETTING OF THE PROBLEM

As in [4] we are going to work only with dyadic martingales that are obtained from the given realizations. However, the reader can note that the results are exactly the same for general case, moreover, the Bellman function is the same.

Consider a probability space

$$\left(I, \mathcal{B}, \frac{dx}{|I|}\right).$$

Let \mathcal{M}_n be the σ -algebra generated by the dyadic intervals

$$I_{n,j} \stackrel{\text{def}}{=} \left[\frac{j|I|}{2^n}, \frac{(j+1)|I|}{2^n} \right), \quad 0 \leq j \leq 2^n - 1.$$

Thus \mathcal{M}_n consists of unions of the dyadic intervals $I_{n,j}$, $I = \mathcal{M}_0$ and $\mathcal{M}_n \subset \mathcal{M}_{n+1}$.

Generally such sequence is called filtration and in our particular case we deal with the dyadic filtration.

For a given \mathbb{R}^m valued function $F \in L^1(I)$ we set

$$F_n \stackrel{\text{def}}{=} \mathbb{E}(F | \mathcal{M}_n) \stackrel{\text{def}}{=} \sum_j \langle F \rangle_{I_{n,j}} \chi_{I_{n,j}}$$

and $\mathbb{E}F \stackrel{\text{def}}{=} \langle F \rangle_I$ where

$$\langle F \rangle_J \stackrel{\text{def}}{=} \frac{1}{|J|} \int_J F(t) dt$$

for any interval J of the real line. We recall that integral of the vectorvalued function is understood componentwise. Note that $F_n = \mathbb{E}F_{n+1} | \mathcal{M}_n$, which is equivalent to the identity

$$(4) \quad \langle F \rangle_{I_{n,j}} = \frac{\langle F \rangle_{I_{n+1,k}} + \langle F \rangle_{I_{n+1,k+1}}}{2}, \text{ where } I_{n,j} = I_{n+1,k} \cup I_{n+1,k+1}.$$

Thus we say that $\{F_n\}_{n=0}^\infty$ is a dyadic martingale constructed by F .

Now we are able to define one of the most extreme version of the martingale transform.

Definition 1. *Let F and G be real valued integrable functions. If the dyadic martingale $\{G_n\}$ constructed by G satisfies $|G_{n+1} - G_n| = |F_{n+1} - F_n|$ for each $n \geq 0$, then G is called martingale transform of F .*

Recall that we are interested in the estimate

$$(5) \quad \|(G^2 + \tau^2 F^2)^{1/2}\|_{L^p} \leq C \|F\|_{L^p}.$$

Remark 2. *The reader noticed that we don't consider estimate 5 for pairs (F_n, G_n) because estimates for such pairs easily follow from the estimate for the limited pairs (F, G) if we set $F = F_n$ and $G = G_n$.*

Let us introduce a Bellman function

$$(6) \quad H(\mathbf{x}) \stackrel{\text{def}}{=} \sup_{F,G} \{ \mathbb{E}B(\varphi(F, G)), \mathbb{E}\varphi(F, G) = \mathbf{x}, |G_{n+1} - G_n| = |F_{n+1} - F_n|, n \geq 0 \}.$$

where $\varphi(x_1, x_2) = (x_1, x_2, |x_1|^p)$, $B(\varphi(x_1, x_2)) = (x_2^2 + \tau^2 x_1^2)^{p/2}$, $\mathbf{x} = (x_1, x_2, x_3)$.

Remark 3. *Further bold lowercase letters mean points in \mathbb{R}^3 .*

Then we see that estimate (5) can be rewritten as follows:

$$H(x_1, x_2, x_3) \leq C^p x_3.$$

We mention that the Bellman function H does not depend on the choice of the interval I . This is easy and common place which the reader can proof himself or see [4], [5]. Further we assume that $I = [0, 1]$.

The following propositions investigate the properties of the Bellman function $H(\mathbf{x})$, and it explains why we look for $H(x_1, x_2, x_3)$ as for the minimal diagonally

concave function on the domain $\Omega = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : |x|^p \leq x_3\}$ with the boundary condition

$$(7) \quad H(x_1, x_2, |x_1|^p) = (x_2^2 + \tau^2 x_1^2)^{p/2}.$$

Definition 2. A pair (F, G) is said to be admissible for the point $\mathbf{x} \in \mathbb{R}^3$ if G is a martingale transform of F and $\mathbb{E}(F, G, |F|^p) = \mathbf{x}$.

Proposition 1. The domain of definition of the Bellman function $H(\mathbf{x})$ is $\Omega = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : |x_1|^p \leq x_3\}$, and the boundary condition is $H(x_1, x_2, |x_1|^p) = (x_2^2 + \tau^2 x_1^2)^{p/2}$.

Proof. First, we check that $\text{Dom} H \subseteq \Omega$. If $\mathbf{x} \in \text{Dom} H$ there exists an admissible pair (F, G) for the point \mathbf{x} i.e. $\mathbf{x} = \mathbb{E}(F, G, |F|^p)$. By Jensen's inequality we have $|\mathbb{E}F|^p \leq \mathbb{E}|F|^p$ and therefore $\mathbf{x} \in \Omega$.

Second, we check that $\Omega \subseteq \text{Dom} H$. For every point $\mathbf{x} \in \Omega$ we need to find an admissible pair (F, G) . If $\mathbf{x} \in \partial\Omega$ then we take the constant functions $F = x_1, G = x_2$ so that $(F, G, |F|^p) = \mathbf{x}$. If \mathbf{x} belongs to the interior of Ω , then we draw the line for example in $x_1 + x_2 = A$ plane for a constant A , such that it intersects the boundary of Ω at two points \mathbf{a}, \mathbf{b} so that $|\mathbf{ax}|/|\mathbf{xb}| = 1$. Note that \mathbf{ax} denotes the segment joining the points \mathbf{a} and \mathbf{x} . We take the constants $F_{\mathbf{a}}, G_{\mathbf{a}}$ admissible to the point \mathbf{a} and constants $F_{\mathbf{b}}, G_{\mathbf{b}}$ that are admissible to the point \mathbf{b} . Finally we consider the concatenation (F, G) of these two pairs $(F_{\mathbf{a}}, G_{\mathbf{a}}), (F_{\mathbf{b}}, G_{\mathbf{b}})$ in the following way:

$$(F(t), G(t)) \stackrel{\text{def}}{=} \begin{cases} (F_{\mathbf{a}}(2t), G_{\mathbf{a}}(2t)) & t \in [0, 1/2] \\ (F_{\mathbf{b}}(2(t - 1/2)), G_{\mathbf{b}}(2(t - 1/2))) & t \in [1/2, 1]; \end{cases}$$

One can easily show that the pair (F, G) is admissible for the point \mathbf{x} .

Finally, we check the boundary condition $H(x_1, x_2, |x_1|^p) = (x_2^2 + \tau^2 x_1^2)^{p/2}$. Condition $\mathbf{x} \in \partial\Omega$ implies that the pair (F, G) admissible for the point \mathbf{x} has the following property: $|\mathbb{E}F|^p = \mathbb{E}|F|^p$ which means that $F = \text{const} = x_1$ and since G is a martingale transform of F we have $|G_{n+1} - G_n| = 0$ for all $n \geq 0$, therefore, $G = \text{const} = x_2$. Last deduction is consequence of the fact that Haar system is complete in L^1 . Thus, in (6) supremum is taken over the one element set (constant function), therefore, we obtain the desired result. \square

Definition 3. A function U defined on Ω is called diagonally concave, if for every constant $A \in \mathbb{R}$ it is a concave function in $\Omega \cap \{(x_1, x_2, x_3) : x_1 + x_2 = A\}$ and in $\Omega \cap \{(x_1, x_2, x_3) : x_1 - x_2 = A\}$.

Proposition 2. $H(\mathbf{x})$ is a diagonally concave function in Ω .

Proof. Firstly, we check that $H(\mathbf{x})$ is concave in the plane $\Omega \cap \{(x_1, x_2, x_3) : x_1 + x_2 = A\}$ for some fixed $A \in \mathbb{R}$. The case of orthogonal planes is similar. Pick two points $\mathbf{x}, \mathbf{y} \in \Omega \subset \mathbb{R}^3$ such that a segment \mathbf{xy} belongs to a plane $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 = A\}$ for a constant A . For any $\varepsilon > 0$, there exists the pair of functions $(F_{\mathbf{x}}, G_{\mathbf{x}})$, where $G_{\mathbf{x}}$ is martingale transform of $F_{\mathbf{x}}$, such that $\mathbb{E}\varphi(F_{\mathbf{x}}, G_{\mathbf{x}}) = \mathbf{x}$ and $H(\mathbf{x}) < \mathbb{E}B(\varphi(F_{\mathbf{x}}, G_{\mathbf{x}})) + \varepsilon$. Similarly, for the point \mathbf{y} we can find the pair of functions $(F_{\mathbf{y}}, G_{\mathbf{y}})$, such that $H(\mathbf{y}) < \mathbb{E}B(\varphi(F_{\mathbf{y}}, G_{\mathbf{y}})) + \varepsilon$. Now we concatenate these

pairs of functions in the following way

$$(F(t), G(t)) \stackrel{\text{def}}{=} \begin{cases} (F_{\mathbf{x}}(2t), G_{\mathbf{x}}(2t)) & t \in [0, 1/2], \\ (F_{\mathbf{y}}(2(t - 1/2)), G_{\mathbf{y}}(2(t - 1/2))) & t \in [1/2, 1]. \end{cases}$$

Then by change of variables in integral we have

$$\begin{aligned} \langle \varphi(F, G) \rangle_{[0,1]} &= \frac{1}{2} \left(\langle \varphi(F, G) \rangle_{[0,1/2]} + \langle \varphi(F, G) \rangle_{[1/2,1]} \right) = \\ &= \frac{1}{2} \langle \varphi(F_{\mathbf{x}}, G_{\mathbf{x}}) \rangle_{[0,1]} + \frac{1}{2} \langle \varphi(F_{\mathbf{y}}, G_{\mathbf{y}}) \rangle_{[0,1]} = \frac{\mathbf{x} + \mathbf{y}}{2}. \end{aligned}$$

Note that G is a martingale transform of F . Indeed, for $n = 0$ the equality $|F_{n+1} - F_n| = |G_{n+1} - G_n|$ is true because the segment \mathbf{xy} lies in the plane we have mentioned. For $n \geq 1$ the above equality is true because it is true for the pairs $(F_{\mathbf{x}}, G_{\mathbf{x}})$ and $(F_{\mathbf{y}}, G_{\mathbf{y}})$. Therefore, similarly as above we have

$$\begin{aligned} H\left(\frac{\mathbf{x} + \mathbf{y}}{2}\right) &\geq \langle B(\varphi(F, G)) \rangle_{[0,1]} = \\ &= \frac{1}{2} \left[\langle B(\varphi(F, G)) \rangle_{[0,1/2]} + \langle B(\varphi(F, G)) \rangle_{[1/2,1]} \right] = \\ &= \frac{1}{2} \langle B(\varphi(F_{\mathbf{x}}, G_{\mathbf{x}})) \rangle_{[0,1]} + \frac{1}{2} \langle B(\varphi(F_{\mathbf{y}}, G_{\mathbf{y}})) \rangle_{[0,1]} > \frac{1}{2} H(\mathbf{x}) + \frac{1}{2} H(\mathbf{y}) - \varepsilon. \end{aligned}$$

So, the rest follows by sending ε to zero. \square

Remark 4. We should mention that the important part in the proposition discussed above was that, firstly, our functional has the type $\mathbb{E}B(\xi)$, secondly, every atom of \mathcal{M}_n can be linearly transformed onto \mathcal{M}_0 . Therefore, type of filtration did not play any role.

Proposition 3. If U is a continuous diagonally concave function in Ω with a boundary condition $U(x_1, x_2, |x_1|^p) \geq (x_2^2 + \tau^2 x_1^2)^{p/2}$, then $U \geq H$ in Ω .

Proof. Indeed, let $\xi = (F, G, |F|^p)$ and let $\xi_n = \mathbb{E}\xi | \mathcal{M}_n$ for all $n \geq 0$. Since on each atom from \mathcal{M}_n the differences $G_{n+1} - G_n$, $F_{n+1} - F_n$ either coincide or have a different sign, therefore by virtue of diagonally concavity of the function U we have $U(\xi_n) - \mathbb{E}U(\xi_{n+1}) | \mathcal{M}_n = U(\xi_n) - \mathbb{E}U(\xi_n + (\xi_{n+1} - \xi_n)) | \mathcal{M}_n \geq 0$. Thus $U(\xi_n)$ is a supermartingale. Because the boundary data of U is nonnegative, then U is nonnegative itself. Indeed, for example, choose the point where U is negative. Draw the line that passes through that point and lies in the plane $\{(x_1, x_2, x_3) : x_1 + x_2 = C\}$ for a constant C so that this line intersects the boundary of Ω at points \mathbf{a}, \mathbf{b} . Because U is concave then at least at one point among \mathbf{a}, \mathbf{b} the function U has negative value which contradicts to the positivity of the boundary condition. So $\eta_n \stackrel{\text{def}}{=} U(\xi_n)$ is a non-negative supermartingale. Hence, there exists almost everywhere limit $\lim_n U(\xi_n)$ and $\mathbb{E} \lim_n U(\xi_n) \leq U(\xi_0)$. On the other hand, $a.e. \lim \xi_n \rightarrow \xi$ (because of Lebesgue differentiation theorem) and U is continuous. Therefore, we have $\mathbb{E}(G^2 + \tau^2 F^2)^{p/2} \leq \mathbb{E}U(\xi) \leq U(\xi_0)$. \square

Before we finish this section we try to explain our strategy of finding the Bellman function H . We believe that our function H is actually minimal diagonally concave function in Ω with the boundary condition $H|_{\partial\Omega} = (x_2^2 + \tau^2 x_1^2)^{p/2}$.

So, we are going to find a minimal candidate B , that is continuous, diagonally concave, with the fixed boundary condition $B|_{\partial\Omega} = (y^2 + \tau^2 x^2)^{p/2}$. We caution the reader that symbol B also denotes boundary data, however, in Section 6 we are going to use symbol B as the candidate for minimal diagonally concave function. Surely $B \geq H$ by virtue of Proposition 3. In the Section 6 we will see that for each point $\mathbf{x} \in \Omega$ and any $\varepsilon > 0$, we can construct an admissible pair (F, G) such that $B(\mathbf{x}) < \mathbb{E}(F^2 + \tau^2 G^2)^{p/2} + \varepsilon$. This will show that $B \leq H$ and hence $B = H$.

Before we begin to do this, we have to elaborate the few preliminaries from the pure differential geometry. In the following section we will talk about homogeneous Monge–Ampère equation and minimal concave functions.

3. HOMOGENEOUS MONGE–AMPÈRE EQUATION AND MINIMAL CONCAVE FUNCTIONS

3.1. Foliation. Let $g(s) \in C^3(I)$ be such that $g'' > 0$, and let Ω be a convex domain which is bounded by the curve $(s, g(s))$ and the tangents that pass through the end-points of the curve (see Figure 1). Fix some function $f(s) \in C^3(I)$. The first

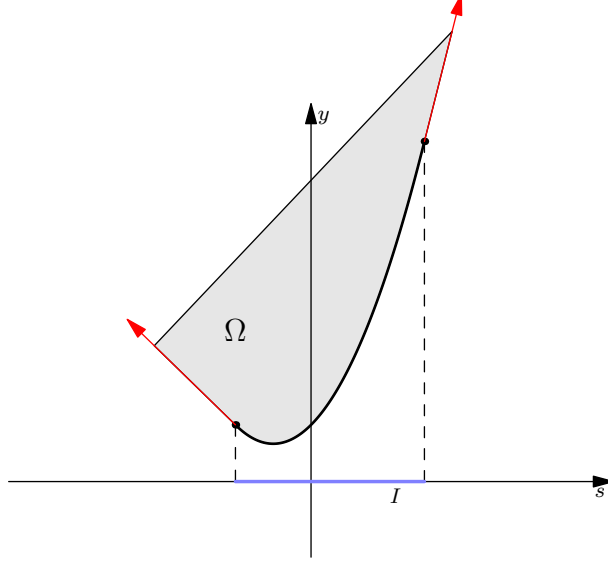


FIGURE 1. Domain Ω

question we ask is the following: how the minimal concave function $\mathbf{B}(x_1, x_2)$ with boundary data $\mathbf{B}(s, g(s)) = f(s)$ looks *locally* in a subdomain of Ω . In other words take a convex hull of the curve $(s, g(s), f(s)), s \in I$ then the question is how the boundary of this convex hull looks like.

We recall that the concavity is equivalent to the following inequalities:

$$(8) \quad \det(d^2\mathbf{B}) \geq 0,$$

$$(9) \quad \mathbf{B}''_{x_1x_1} + \mathbf{B}''_{x_2x_2} \leq 0.$$

Expression 8 is the Gaussian curvature of the surface $(x_1, x_2, \mathbf{B}(x_1, x_2))$ up to a positive factor $(1 + (\mathbf{B}'_{x_1})^2 + (\mathbf{B}'_{x_2})^2)^2$. So in order to make the surface minimal, it is reasonable to minimize the Gaussian curvature. Therefore, we will look for a surface with the zero Gaussian curvature. And here arises the homogeneous Monge–Ampère equation $\det(d^2\mathbf{B}) = 0$. These surfaces are known as developable surfaces i.e. such a surface can be constructed by bending a plane region. The important property of such surfaces is that they consist by line segments i.e. function \mathbf{B} satisfying homogeneous Monge–Ampère equation $\det(d^2\mathbf{B}) = 0$ is linear along some *family of segments*. These considerations lead us to investigate such functions \mathbf{B} . Firstly, we define a *foliation*. For any segment ℓ in the Euclidian space by symbol ℓ° we mean an open segment i.e. ℓ without endpoints.

Fix any subinterval $J \subseteq I$. By symbol $\Theta(J, g)$ we denote an arbitrary set of nontrivial segments (i.e. single points are excluded) in \mathbb{R}^2 with the following requirements:

1. For any $\ell \in \Theta(J, g)$ we have $\ell^\circ \in \Omega$.
2. For any $\ell_1, \ell_2 \in \Theta(J, g)$ we have $\ell_1 \cap \ell_2 = \emptyset$.
3. For any $\ell \in \Theta(J, g)$ there exists only one point $s \in J$ such that $(s, g(s))$ is one of the end point of the segment ℓ and vice versa, for any point $s \in J$ there exists $\ell \in \Theta(J, g)$ such that $(s, g(s))$ is one of the end point of the segment ℓ .
4. There exists C^1 smooth argument function $\theta(s)$.

We explain the meaning of requirement 4. To each point $s \in J$ there corresponds only one segment $\ell \in \Theta(J, g)$ with an endpoint $(s, g(s))$. Take a nonzero vector with initial point at $(s, g(s))$, parallel to the segment ℓ and having an endpoint in Ω . We define the value of $\theta(s)$ to be an argument of this vector. Surely argument is defined up to additive number $2\pi k$ where $k \in \mathbb{Z}$. Nevertheless we take any representative from these angles. Similarly we do for all other points $s \in I$. By this way we get family of functions $\theta(s)$. If there exists $C^1(J)$ smooth function $\theta(s)$ from this family then the requirement 4 is satisfied.

Remark 5. *It is clear that if $\theta(s)$ is $C^1(J)$ smooth argument function, then for any $k \in \mathbb{Z}$, $\theta(s) + 2\pi k$ is also $C^1(J)$ smooth argument function. Any two $C^1(J)$ smooth argument functions differ by constant $2\pi n$ for some $n \in \mathbb{Z}$.*

This remark is consequence of the fact that the quantity $\theta'(s)$ is well defined regardless of the choices of $\theta(s)$.

Next we define $\Omega(\Theta(J, g)) = \bigcup_{\ell \in \Theta(J, g)} \ell^\circ$. Given a point $x \in \Omega(\Theta(J, g))$ we denote by symbol $\ell(x)$ a segment $\ell(x) \in \Theta(J, g)$ which passes through the point x . If $x = (s, g(s))$ then instead of $\ell((s, g(s)))$ we just write $\ell(s)$. Surely such a segment exists, and it is unique. We denote by symbol $s(x)$ a point $s(x) \in J$ such that $(s(x), g(s(x)))$ is one of the end point of the segment $\ell(x)$. Moreover, in a natural

way we set $s(x) = s$ if $x = (s, g(s))$. It is clear that such $s(x)$ exists, and it is unique. We introduce a function

$$(10) \quad K(s) = g'(s) \cos \theta(s) - \sin \theta(s), \quad s \in J.$$

Note that that $K < 0$. This inequality becomes obvious if we rewrite $g'(s) \cos \theta(s) - \sin \theta(s) = \langle (1, g'), (-\sin \theta, \cos \theta) \rangle$ and take into account requirement 1 of $\Theta(J, g)$. Note that $\langle \cdot, \cdot \rangle$ means scalar product in Euclidian space.

We need few more requirements on $\Theta(J, g)$.

5. For any $x = (x_1, x_2) \in \Omega(\Theta(J, g))$ we have an inequality

$$K(s(x)) + \theta'(s(x)) \|(x_1 - s(x), x_2 - g(s(x)))\| < 0.$$

6. The function $s(x)$ is continuous function in $\Omega(\Theta(J, g)) \cup \Gamma(J)$ where $\Gamma(J) = \{(s, g(s)) : s \in J\}$.

Note that if $\theta'(s) \leq 0$ (which happens in most of the cases) then the requirement 5 holds. If we know the endpoints of the segments $\Theta(J, g)$, then it is easy to see that requirement 5 is necessary and sufficient to check at those points $x = (x_1, x_2)$, where x is the another endpoint of the segment other than $(s, g(s))$. Roughly speaking the requirement 5 means the segments of $\Theta(J, g)$ do not rotate rapidly counterclockwise.

Definition 4. Set of segments $\Theta(J, g)$ with the requirements mentioned above is called foliation. The set $\Omega(\Theta(J, g))$ is called domain of foliation.

A typical example of a foliation is given on Figure 2.

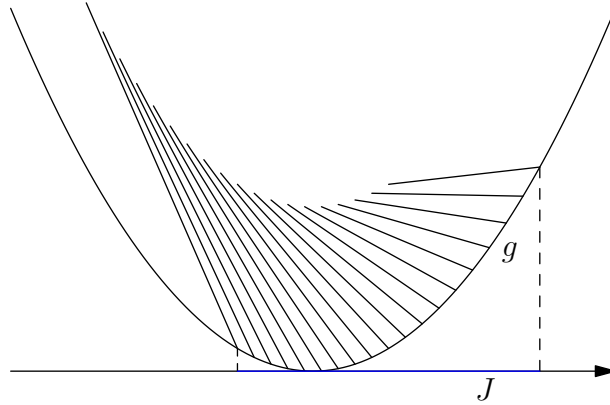


FIGURE 2. Foliation $\Omega(J, g)$

Lemma 1. The function $s(x)$ belongs to $C^1(\Omega(\Theta(J, g)))$. Moreover

$$(11) \quad (s'_{x_1}, s'_{x_2}) = \frac{(\sin \theta, -\cos \theta)}{-K(s) - \theta' \cdot \|(x_1 - s, x_2 - g(s))\|}.$$

Proof. Definition of the function $s(x)$ implies that

$$-(x_1 - s) \sin \theta(s) + (x_2 - g(s)) \cos \theta(s) = 0.$$

Therefore the lemma is immediate consequence of the implicit function theorem. \square

Let $J = [s_1, s_2] \subseteq I$, and let $(s, g(s), f(s)) \in C^3(I)$ be such that $g'' > 0$ on I . Consider an arbitrary foliation $\Theta(J, g)$ with an arbitrary $C^1([s_1, s_2])$ smooth argument function $\theta(s)$. We need the following technical lemma.

Lemma 2. *Solutions of the system of equations*

$$(12) \quad t_1'(s) \cos \theta(s) + t_2'(s) \sin \theta(s) = 0,$$

$$(13) \quad t_1(s) + t_2(s)g'(s) = f'(s), \quad s \in J$$

are the following functions

$$\begin{aligned} t_1(s) &= \int_{s_1}^s \left(\frac{g''(r)}{K(r)} \sin \theta(r) \cdot t_2(r) - \frac{f''(r)}{K(r)} \sin \theta(r) \right) dr + f'(s_1) - t_2(s_1)g'(s_1), \\ t_2(s) &= t_2(s_1) \exp \left(- \int_{s_1}^s \frac{g''(r)}{K(r)} \cos \theta(r) dr \right) + \\ &\quad \int_{s_1}^s \frac{f''(y)}{K(y)} \exp \left(- \int_y^s \frac{g''(r)}{K(r)} \cos \theta(r) dr \right) \cos \theta(y) dy, \quad s \in J \end{aligned}$$

where $t_2(s_1)$ is an arbitrary real number.

Proof. We differentiate equality (13), after that the system takes the following form

$$\begin{pmatrix} \cos \theta & \sin \theta \\ 1 & g' \end{pmatrix} \begin{pmatrix} t_1' \\ t_2' \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -g'' \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} + \begin{pmatrix} 0 \\ f'' \end{pmatrix}.$$

This implies that

$$(14) \quad \begin{pmatrix} t_1' \\ t_2' \end{pmatrix} = \frac{g''}{K} \begin{pmatrix} 0 & \sin \theta \\ 0 & -\cos \theta \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} + \frac{f''}{K} \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix}.$$

By solving this system of differential equations and using the fact that $t_1(s_1) + g'(s_1)t_2(s_1) = f'(s_1)$ we get the desired result. \square

Remark 6. *Integration by parts allows us to rewrite the expression of $t_2(s)$ in the following form*

$$\begin{aligned} t_2(s) &= \exp \left(- \int_{s_1}^s \frac{g''(r)}{K(r)} \cos \theta(r) dr \right) \left(t_2(s_1) - \frac{f''(s_1)}{g''(s_1)} \right) + \frac{f''(s)}{g''(s)} - \\ &\quad - \int_{s_1}^s \left[\frac{f''(y)}{g''(y)} \right]' \exp \left(- \int_y^s \frac{g''(r)}{K(r)} \cos \theta(r) dr \right) dy. \end{aligned}$$

Definition 5. *We say that a function \mathbf{B} has a foliation $\Theta(J, g)$ if it is continuous on $\Omega(\Theta(J, g))$, and it is linear on each segment of $\Theta(J, g)$.*

The following lemma describes how to construct a function \mathbf{B} with a given foliation $\Theta(J, g)$, boundary condition $\mathbf{B}(s, g(s)) = f(s)$ and such that \mathbf{B} satisfies homogeneous Monge–Ampère equation.

Consider a function \mathbf{B} defined by the following way

$$(15) \quad \mathbf{B}(x) = f(s) + \langle t(s), x - (s, g(s)) \rangle, \quad x = (x_1, x_2) \in \Omega(\Theta(J, g))$$

where $s = s(x)$, and $t(s) = (t_1(s), t_2(s))$ satisfies the system of equations 12, 13 with an arbitrary $t_2(s_1)$.

Lemma 3. *The function \mathbf{B} defined by (15) satisfies the following properties:*

1. $\mathbf{B} \in C^2(\Omega(\Theta(J, g))) \cap C^1(\Omega(\Theta(J, g)) \cup \Gamma)$, \mathbf{B} has a foliation $\Theta(J, g)$ and
- (16) $\mathbf{B}(s, g(s)) = f(s) \quad \text{for all } s \in [s_1, s_2].$
2. $\nabla \mathbf{B}(x) = t(s)$ where $s = s(x)$, moreover \mathbf{B} satisfies homogeneous Monge–Ampère equation.

Implicitly this result is given in [7], which is consequence of the Pogorelov’s results about the solution of the homogeneous Monge–Ampère equation. The only difference is that we require for \mathbf{B} to have a given foliation.

Proof. The fact that \mathbf{B} has the foliation $\Theta(J, g)$, and it satisfies equality 16 immediately follows from the definition of the function \mathbf{B} . We check condition of smoothness. By Lemma 1 and Lemma 2 we have $s(x) \in C^2(\Omega(\Theta(J, g)))$ and $t_1, t_2 \in C^1(J)$, therefore the right-hand side of (15) is differentiable with respect to x . So after differentiation of (15) we get

$$(17) \quad \begin{aligned} \nabla \mathbf{B}(x) &= [f'(s) - \langle t(s), (1, g'(s)) \rangle] (s'_{x_1}, s'_{x_2}) + \\ &+ t(s) + \langle t'(s), x - (s, g(s)) \rangle (s'_{x_1}, s'_{x_2}). \end{aligned}$$

Using (12) and (13) we obtain $\nabla \mathbf{B}(x) = t(s)$. Taking derivative with respect to x second times we get

$$\begin{aligned} \frac{\partial^2 \mathbf{B}}{\partial x_1^2} &= t'_1(s) s'_{x_1}, \quad \frac{\partial^2 \mathbf{B}}{\partial x_2 \partial x_1} = t'_1(s) s'_{x_2}, \\ \frac{\partial^2 \mathbf{B}}{\partial x_1 \partial x_2} &= t'_2(s) s'_{x_1}, \quad \frac{\partial^2 \mathbf{B}}{\partial x_2^2} = t'_2(s) s'_{x_2}. \end{aligned}$$

Using (12) we get that $t'_1(s) s'_{x_2} = t'_2(s) s'_{x_1}$, therefore $\mathbf{B} \in C^2(\Omega(\Theta(J, g)))$. Finally, we check that \mathbf{B} satisfies homogeneous Monge–Ampère equation. Indeed,

$$\begin{aligned} \det(d^2 \mathbf{B}) &= \frac{\partial^2 \mathbf{B}}{\partial x_1^2} \cdot \frac{\partial^2 \mathbf{B}}{\partial x_2^2} - \frac{\partial^2 \mathbf{B}}{\partial x_2 \partial x_1} \cdot \frac{\partial^2 \mathbf{B}}{\partial x_1 \partial x_2} = \\ &= t'_1(s) s'_{x_1} \cdot t'_2(s) s'_{x_2} - t'_1(s) s'_{x_2} \cdot t'_2(s) s'_{x_1} = 0. \end{aligned}$$

□

Definition 6. *The function $t(s) = (t_1(s), t_2(s)) = \nabla \mathbf{B}(x)$, $s = s(x)$ is called gradient function corresponding to \mathbf{B} .*

The following lemma investigates concavity of the function \mathbf{B} defined by (15). Let $\|\tilde{\ell}(x)\| = \|(s(x) - x_1, g(s(x)) - x_2)\|$ where $x = (x_1, x_2) \in \Omega(\Theta(J, g))$.

Lemma 4. *The following equalities hold*

$$\begin{aligned} \frac{\partial^2 \mathbf{B}}{\partial x_1^2} + \frac{\partial^2 \mathbf{B}}{\partial x_2^2} &= \frac{g''}{K(K + \theta' \|\tilde{\ell}(x)\|)} \left(-t_2 + \frac{f''}{g''} \right) = \\ &= \frac{g''}{K(K + \theta' \|\tilde{\ell}(x)\|)} \times \left[-\exp \left(-\int_{s_1}^s \frac{g''(r)}{K(r)} \cos \theta(r) dr \right) \left(t_2(s_1) - \frac{f''(s_1)}{g''(s_1)} \right) \right. \\ &\quad \left. + \int_{s_1}^s \left[\frac{f''(y)}{g''(y)} \right]' \exp \left(-\int_y^s \frac{g''(r)}{K(r)} \cos \theta(r) dr \right) dy \right]. \end{aligned}$$

Proof. Note that

$$\frac{\partial^2 \mathbf{B}}{\partial x_1^2} + \frac{\partial^2 \mathbf{B}}{\partial x_2^2} = t_1'(s)s_1' + t_2'(s)s_2'.$$

Therefore the lemma is direct computation and application of equalities (11), (12), (13) and Remark 6. \square

Finally, we get one of the important corollary of the current section.

Corollary 1. *Function \mathbf{B} is concave in $\Omega(\Theta(J, g))$ if and only if $\mathcal{F}(s) \leq 0$, where*

$$\begin{aligned} (18) \quad \mathcal{F}(s) &= -\exp \left(-\int_{s_1}^s \frac{g''(r)}{K(r)} \cos \theta(r) dr \right) \left(t_2(s_1) - \frac{f''(s_1)}{g''(s_1)} \right) \\ &\quad + \int_{s_1}^s \left[\frac{f''(y)}{g''(y)} \right]' \exp \left(-\int_y^s \frac{g''(r)}{K(r)} \cos \theta(r) dr \right) dy = \frac{f''(s)}{g''(s)} - t_2(s). \end{aligned}$$

Proof. \mathbf{B} satisfies homogeneous Monge–Ampère equation. Therefore \mathbf{B} is concave if and only if

$$(19) \quad \frac{\partial^2 \mathbf{B}}{\partial x_1^2} + \frac{\partial^2 \mathbf{B}}{\partial x_2^2} \leq 0.$$

Note that

$$\frac{g''}{K(K + \theta' \|\tilde{\ell}(x)\|)} > 0.$$

Hence, by virtue of Lemma 4, inequality (19) holds if and only if $\mathcal{F}(s) \leq 0$. \square

Further the function \mathcal{F} will be called *force function*.

Remark 7. *The fact $t_2(s) = f''/g'' - \mathcal{F}$ with (14) implies that the force function \mathcal{F} satisfies the following differential equation*

$$\begin{aligned} (20) \quad \mathcal{F}' + \mathcal{F} \cdot \frac{\cos \theta}{K} g'' - \left[\frac{f''}{g''} \right]' &= 0, \quad s \in J \\ \mathcal{F}(s_1) &= \frac{f''(s_1)}{g''(s_1)} - t_2(s_1). \end{aligned}$$

We remind the reader that for an arbitrary smooth curve $\gamma = (s, g(s), f(s))$ torsion has the following expression

$$\frac{\det(\gamma', \gamma'', \gamma''')}{\|\gamma' \times \gamma''\|^2} = \frac{f'''g'' - g'''f''}{\|\gamma' \times \gamma''\|^2} = \frac{(g'')^2}{\|\gamma' \times \gamma''\|^2} \cdot \left[\frac{f''}{g''} \right]'.$$

Corollary 2. *If $\mathcal{F}(s_1) \leq 0$ and torsion of a curve $(s, g(s), f(s))$, $s \in J$ is negative then the function \mathbf{B} defined by (15) is concave.*

Proof. The corollary is immediate consequence of (18). \square

Thus, we see that the torsion of the boundary data plays the crucial role in concavity of the surface with zero Gaussian curvature. More detailed investigations about how we choose the constant $t_2(s_1)$ will be given in the Section 3.2.

Let $\Theta(J, g)$ and $\tilde{\Theta}(J, g)$ be foliations with some argument functions $\theta(s)$ and $\tilde{\theta}(s)$ respectively. Let \mathbf{B} and $\tilde{\mathbf{B}}$ be corresponding functions defined by (15), and let $\mathcal{F}, \tilde{\mathcal{F}}$ be corresponding force functions. Note that equality $\mathcal{F}(s) = \tilde{\mathcal{F}}(s)$ is equivalent to the equality $t(s) = \tilde{t}(s)$ where $t(s) = (t_1(s), t_2(s))$ and $\tilde{t}(s) = (\tilde{t}_1(s), \tilde{t}_2(s))$ are corresponding gradients of \mathbf{B} and $\tilde{\mathbf{B}}$ (see (13) and Corollary 1).

Assume that the functions \mathbf{B} and $\tilde{\mathbf{B}}$ are concave functions.

Lemma 5. *If $\sin(\tilde{\theta} - \theta) \geq 0$ for all $s \in J$, and $\mathcal{F}(s_1) = \tilde{\mathcal{F}}(s_1)$ then $\tilde{\mathbf{B}} \leq \mathbf{B}$ on $\Omega(\Theta(J, g)) \cap \tilde{\Omega}(\tilde{\Theta}(J, g))$.*

In other words the lemma says that if at initial point $(s_1, g(s_1))$ gradients of the functions $\tilde{\mathbf{B}}$ and \mathbf{B} coincide, however the foliation $\tilde{\Theta}(J, g)$ is “to the left of” the foliation $\Theta(J, g)$ (see Figure 3) then $\tilde{\mathbf{B}} \leq \mathbf{B}$ provided \mathbf{B} and $\tilde{\mathbf{B}}$ are concave.

Proof. Let K and \tilde{K} be a corresponding functions of \mathbf{B} and $\tilde{\mathbf{B}}$ defined by (10). Condition $K, \tilde{K} < 0$ implies that the inequality $\sin(\tilde{\theta} - \theta) \geq 0$ is equivalent to the inequality

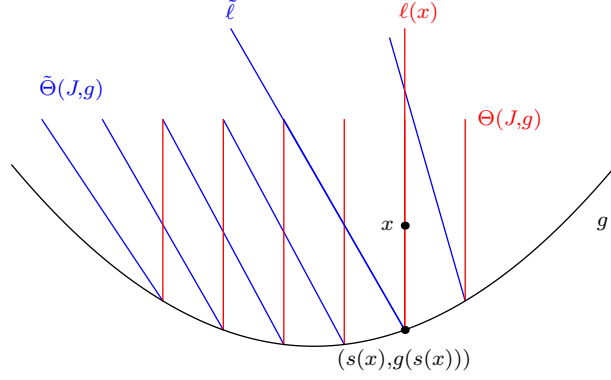
$$(21) \quad \frac{\cos \tilde{\theta}}{\tilde{K}} \leq \frac{\cos \theta}{K} \quad \text{for } s \in J.$$

Indeed, if we rewrite (21) as $K \cos \tilde{\theta} \geq \tilde{K} \cos \theta$ then this simplifies to $-\cos \theta \sin \tilde{\theta} \geq -\cos \theta \sin \tilde{\theta}$, so the result follows. The force functions $\mathcal{F}, \tilde{\mathcal{F}}$ satisfy the differential equation (20) with the same boundary condition $\mathcal{F}(s_1) = \tilde{\mathcal{F}}(s_1)$. Then by (21) and by comparison theorems we get $\tilde{\mathcal{F}} \geq \mathcal{F}$ on J . This and (18) imply that $\tilde{t}_2 \leq t_2$ on J . Pick any point $x \in \Omega(\Theta(J, g)) \cap \tilde{\Omega}(\tilde{\Theta}(J, g))$. Then there exists a segment $\ell(x) \in \Theta(J, g)$. Let $(s(x), g(s(x)))$ be a corresponding endpoint of this segment. There exists a segment $\tilde{\ell} \in \tilde{\Theta}(J, g)$ which has $(s(x), g(s(x)))$ as an endpoint (see Figure 3).

Consider a tangent plane $L(x)$ to $(x_1, x_2, \tilde{\mathbf{B}})$ at point $(s(x), g(s(x)))$. The fact that gradient of $\tilde{\mathbf{B}}$ is constant on $\tilde{\ell}$, implies that L is tangent to $(x_1, x_2, \tilde{\mathbf{B}})$ on $\tilde{\ell}$. Therefore

$$L(x) = f(s) + \langle (\tilde{t}_1(s), \tilde{t}_2(s)), (x_1 - s, x_2 - g(s)) \rangle,$$

where $x = (x_1, x_2)$ and $s = s(x)$. Concavity of $\tilde{\mathbf{B}}$ implies that a value of the function $\tilde{\mathbf{B}}$ at point y seen from the point $(s(x), g(s(x)))$ is less than $L(y)$. In particular

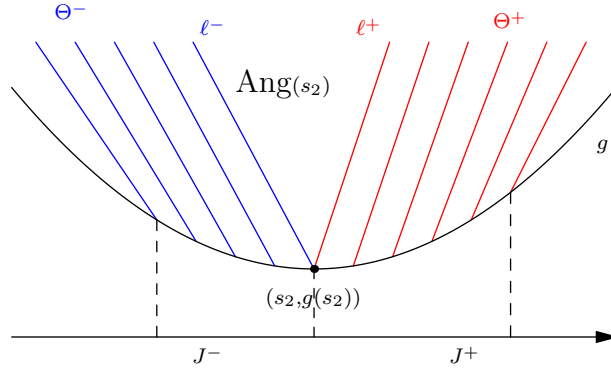
FIGURE 3. Foliations $\Theta(J, g)$ and $\tilde{\Theta}(J, g)$

$\tilde{\mathbf{B}}(x) \leq L(x)$. Now it is enough to prove that $L(x) \leq \mathbf{B}(x)$. By (15) we have

$$\mathbf{B}(x) = f(s) + \langle (t_1(s), t_2(s)), (x_1 - s(x), x_2 - g(s)) \rangle.$$

Therefore using (13), $\langle (-g', 1), (x_1 - s, x_2 - g(s)) \rangle \geq 0$ and the fact that $\tilde{t}_2 \leq t_2$ we get the desired result. \square

Let $J^- = [s_1, s_2]$ and $J^+ = [s_2, s_3]$ where $J^-, J^+ \subset I$. Consider an arbitrary foliations $\Theta^- = \Theta^-(J^-, g)$ and $\Theta^+ = \Theta^+(J^+, g)$ such that $\Omega(\Theta^-) \cap \Omega(\Theta^+) = \emptyset$, and let θ^- and θ^+ be a corresponding argument functions. Let \mathbf{B}^- and \mathbf{B}^+ be a corresponding functions defined by (15), and let $t^- = (t_1^-, t_2^-)$, $t^+ = (t_1^+, t_2^+)$ be a corresponding gradient functions. Set $\text{Ang}(s_2)$ to be a convex hull of $\ell^-(s_2)$ and $\ell^+(s_2)$ where $\ell^-(s_2) \in \Theta^-$, $\ell^+(s_2) \in \Theta^+$ are the segments with the endpoint $(s_2, g(s_2))$ (see Figure 4). We require that $\text{Ang}(s_2) \cap \Omega(\Theta^-) = \ell^-$ and $\text{Ang}(s_2) \cap \Omega(\Theta^+) = \ell^+$.

FIGURE 4. Gluing of \mathbf{B}^- and \mathbf{B}^+

Let $\mathcal{F}^-, \mathcal{F}^+$ be the corresponding forces, and let \mathbf{B}_{Ang} be a function defined linearly on $\text{Ang}(s_2)$ via the values of \mathbf{B}^- and \mathbf{B}^+ on ℓ^-, ℓ^+ respectively.

Lemma 6. *If $t_2^-(s_2) = t_2^+(s_2)$, then the function \mathbf{B} defined by the following formula*

$$\mathbf{B}(x) = \begin{cases} \mathbf{B}^-(x), & x \in \Omega(\Theta(J^-, g)), \\ \mathbf{B}_{\text{Ang}}(x), & x \in \text{Ang}(s_2), \\ \mathbf{B}^+(x), & x \in \Omega(\Theta(J^+, g)), \end{cases}$$

belongs to the class $C^1(\Omega(\Theta^-) \cup \text{Ang}(s_2) \cup \Omega(\Theta^+) \cup \Gamma(J^- \cup J^+))$.

Proof. By (13) condition $t_2^-(s_2) = t_2^+(s_2)$ is equivalent to the condition $t^-(s_2) = t^+(s_2)$. We recall that gradient of \mathbf{B}^- is constant on $\ell^-(s_2)$, and the gradient of \mathbf{B}^+ is constant on $\ell^+(s_2)$, therefore the lemma follows immediately from the fact that $\mathbf{B}^-(s_2, g(s_2)) = \mathbf{B}^+(s_2, g(s_2))$. \square

Remark 8. *The fact $\mathbf{B} \in C^1$ implies that its gradient function $t(s) = \nabla \mathbf{B}$ is well defined, and it is continuous. Unfortunately, it is not necessarily true that $t(s) \in C^1([s_1, s_3])$. However, it is clear that $t(s) \in C^1([s_1, s_2])$, and $t(s) \in C^1([s_2, s_3])$.*

Finally we finish this section by the following important corollary.

Corollary 3. *If in the conditions of Lemma 6 in addition \mathbf{B}^- is concave and a torsion of the curve $(s, g(s), f(s))$ is negative on $[s_2, s_3]$ then the function \mathbf{B} is concave.*

Proof. By Lemma 1 concavity of \mathbf{B}^- implies $\mathcal{F}^-(s_2) \leq 0$. By (18) condition $t_2^-(s_2) = t_2^+(s_2)$ is equivalent to $\mathcal{F}^-(s_2) = \mathcal{F}^+(s_2)$. By Corollary 2 we get that \mathbf{B}^+ is concave. Thus, concavity of \mathbf{B} follows from Lemma 6. \square

3.2. Cup. In this section we are going to consider special type of foliations which is called *Cup*. Fix an interval I and consider an arbitrary curve $(s, g(s), f(s)) \in C^3(I)$. We suppose that $g'' > 0$ on I . Let $a(s) \in C^1(J)$ be a function such that $a'(s) < 0$ on J , where $J = [s_0, s_1]$ is a subinterval of I . Assume that $a(s_0) < s_0$ and $[a(s_1), a(s_0)] \subset I$. Consider a set of open segments $\Theta_{\text{cup}}(J, g)$ consisting of those segments $\ell(s, g(s))$, $s \in J$ such that $\ell(s, g(s))$ is a segment on the plane joining the points $(s, g(s))$ and $(a(s), g(a(s)))$ (see Figure 5).

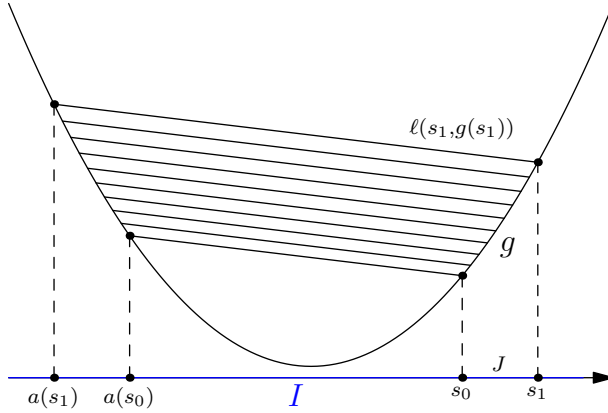


FIGURE 5. Foliation $\Theta_{\text{cup}}(J, g)$

Lemma 7. *The set of segments $\Theta_{\text{cup}}(J, g)$ described above form a foliation.*

Proof. We need to check 6 requirements of foliation. Most of them are trivial except of 4 and 5. We know the endpoints of each segment therefore we can consider the following argument function

$$\theta(s) = \pi + \arctan \left(\frac{g(s) - g(a(s))}{s - a(s)} \right).$$

Surely $\theta(s) \in C^1(J)$, so requirement 4 is satisfied. We check requirement 5. It is clear that it is enough to check this requirement for $x = (a(s), g(a(s)))$. Let $s = s(x)$, then

$$\begin{aligned} K(s) + \theta'(s) \|(a(s) - s, g(a(s)) - g(s))\| &= \frac{\langle (1, g'), (g - g(a), a - s) \rangle}{\|(g(a) - g, s - a)\|} + \\ &= \frac{(g' - a'g'(a))(s - a) - (1 - a')(g - g(a))}{\|(g(a) - g, s - a)\|} = \frac{a' \cdot \langle (1, g'(a)), (g - g(a), a - s) \rangle}{\|(g(a) - g, s - a)\|} \end{aligned}$$

which is strictly negative. \square

Let $\gamma(t) = (t, g(t), f(t)) \in C^3([a_0, b_0])$ be an arbitrary curve such that $g'' > 0$ on $[a_0, b_0]$. Assume that torsion of γ is positive on $I^- = (a_0, c)$, and it is negative on $I^+ = (c, b_0)$ for some $c \in (a_0, b_0)$.

Lemma 8. *For all P such that $0 < P < \min\{c - a_0, b_0 - c\}$ there exist $a \in I^-$, $b \in I^+$ such that $b - a = P$ and*

$$(22) \quad \begin{vmatrix} 1 & 1 & a - b \\ g'(a) & g'(b) & g(a) - g(b) \\ f'(a) & f'(b) & f(a) - f(b) \end{vmatrix} = 0.$$

Proof. Pick a number $a \in (a_0, b_0)$ so that $b = a + P \in (a_0, b_0)$. We denote

$$\mathcal{M}(a, b) = (a - b)(g'(b) - g'(a)) \left(\frac{g(a) - g(b)}{a - b} - g'(a) \right).$$

Note that the conditions $a \neq b$ and $g'' > 0$ imply $\mathcal{M}(a, b) \neq 0$. Then

$$\begin{aligned} &\begin{vmatrix} 1 & 1 & a - b \\ g'(a) & g'(b) & g(a) - g(b) \\ f'(a) & f'(b) & f(a) - f(b) \end{vmatrix} = \\ &\mathcal{M}(a, b) \left[\frac{f(a) - f(b) - f'(a)(a - b)}{g(a) - g(b) - g'(a)(a - b)} - \frac{f'(b) - f'(a)}{g'(b) - g'(a)} \right]. \end{aligned}$$

Thus our equation (22) turns into

$$(23) \quad \frac{f(a) - f(b) - f'(a)(a - b)}{g(a) - g(b) - g'(a)(a - b)} - \frac{f'(b) - f'(a)}{g'(b) - g'(a)} = 0.$$

We consider the following functions $V(x) = f(x) - f'(a)x$ and $U(x) = g(x) - g'(a)x$. Note that $U(a) \neq U(b)$ and $U' \neq 0$ on (a, b) . Therefore by Cauchy's mean

value theorem there exists a point $\xi = \xi(a, b) \in (a, b)$ such that

$$\frac{f(a) - f(b) - f'(a)(a - b)}{g(a) - g(b) - g'(a)(a - b)} = \frac{V(a) - V(b)}{U(a) - U(b)} = \frac{V'(\xi)}{U'(\xi)} = \frac{f'(\xi) - f'(a)}{g'(\xi) - g'(a)}.$$

Now we define

$$W_a(z) \stackrel{\text{def}}{=} \frac{f'(z) - f'(a)}{g'(z) - g'(a)}, \quad z \in (a, b].$$

So the left hand side of (23) takes the following form $W_a(\xi) - W_a(b) = 0$ for some $\xi(a, P) \in (a, b)$. We consider the curve $v(s) = (g'(s), f'(s))$ which is a graph on $[a_0, b_0]$. The fact that the torsion of the curve $\gamma(s) = (s, g(s), f(s))$ changes sign from $+$ to $-$ at point $c \in (a_0, b_0)$ means that the curve $v(s)$ is strictly convex on the interval (a_0, c) , and it is strictly concave on the interval (c, b_0) . We consider a function obtained from (23)

$$(24) \quad D(z) \stackrel{\text{def}}{=} \frac{f(z) - f(z+P) + f'(z)P}{g(z) - g(z+P) + g'(z)P} - \frac{f'(z+P) - f'(z)}{g'(z+P) - g'(z)}, \quad z \in [a_0, c].$$

Note that $D(a_0) = W_{a_0}(\zeta) - W_{a_0}(a_0 + P)$ for some $\zeta = \zeta(a_0, P) \in (a_0, a_0 + P)$. We know that $v(s)$ is strictly convex on the interval $(a_0, a_0 + P)$. This implies that $W_{a_0}(z) - W_{a_0}(a_0 + P) < 0$ for all $z \in (a_0, a_0 + P)$. In particular $D(a_0) < 0$. Similarly, concavity of $v(s)$ on $(c, c + P)$ implies that $D(c) > 0$. Hence, there exists $a \in (a_0, c)$ such that $D(a) = 0$. \square

Let a_1 and b_1 be some solutions of (22) obtained by Lemma 8.

Lemma 9. *There exists a function $a(s) \in C^1((c, b_1]) \cap C([c, b_1])$ such that $a(b_1) = a_1$, $a(c) = c$, $a'(s) < 0$, and the pair $(a(s), s)$ solves the equation (22) for all $s \in [c, b_1]$.*

Proof. Proof of the lemma is consequence of the implicit function theorem. Let $a < b$, and we consider the function

$$\Phi(a, b) \stackrel{\text{def}}{=} \begin{vmatrix} 1 & 1 & a - b \\ g'(a) & g'(b) & g(a) - g(b) \\ f'(a) & f'(b) & f(a) - f(b) \end{vmatrix}.$$

We are going to find the signs of the partial derivatives of $\Phi(a, b)$ at point $(a, b) = (a_1, b_1)$. We present the calculation only for $\partial\Phi/\partial b$. The case for $\partial\Phi/\partial a$ is similar.

$$\begin{aligned} \frac{\partial\Phi(a, b)}{\partial b} &= \begin{vmatrix} 1 & 0 & a - b \\ g'(a) & g''(b) & g(a) - g(b) \\ f'(a) & f''(b) & f(a) - f(b) \end{vmatrix} = \\ &= (a - b)g''(b) \left(\frac{g(a) - g(b)}{a - b} - g'(a) \right) \left[\frac{f(a) - f(b) - f'(a)(a - b)}{g(a) - g(b) - g'(a)(a - b)} - \frac{f''(b)}{g''(b)} \right]. \end{aligned}$$

Note that

$$(a-b)g''(b) \left(\frac{g(a)-g(b)}{a-b} - g'(a) \right) < 0,$$

therefore we see that the sign of $\partial\Phi/\partial b$ depends only on the sign of the expression

$$(25) \quad \frac{f(a)-f(b)-f'(a)(a-b)}{g(a)-g(b)-g'(a)(a-b)} - \frac{f''(b)}{g''(b)}.$$

We use the *cup equation* (23), and we obtain that the expression (25) at point $(a,b) = (a_1,b_1)$ takes the following form:

$$(26) \quad \frac{f'(b)-f'(a)}{g'(b)-g'(a)} - \frac{f''(b)}{g''(b)}$$

The above expression has the following geometrical meaning. We consider the curve $v(s) = (g'(s), f'(s))$, and we draw a segment, which connects the points $v(a)$ and $v(b)$. The above expression is the difference between the slopes of the line which passes through the segment $[v(a), v(b)]$ and tangent line of the curve $v(s)$ at the point b . In the case as it is shown on Figure 6, this difference is positive.

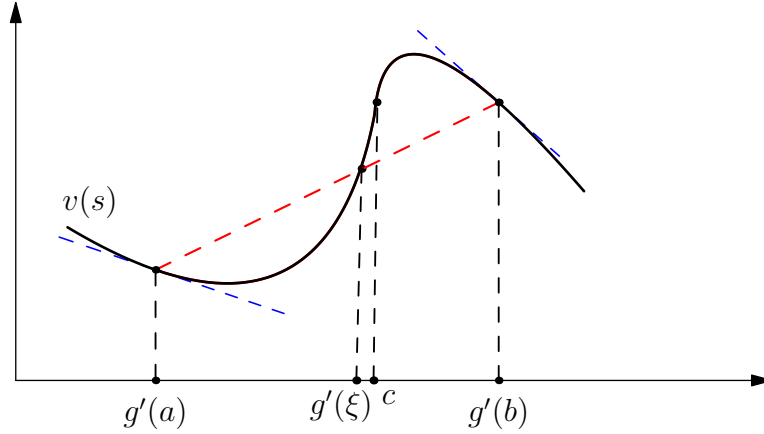


FIGURE 6. Graph $v(s)$

Recall that $v(s)$ is strictly convex on (a_1, c) , and it is strictly concave on (c, b_1) . Therefore, one can easily note that this expression (26) is always positive if the segment $[v(a), v(b)]$ also intersects the curve $v(s)$ at a point ξ such that $a < \xi < b$. This always happens in our case because equation (23) means that the points $v(a), v(\xi), v(b)$ lie on the same line, where ξ was determined from the Cauchy's mean value theorem. Thus

$$(27) \quad \frac{f'(b)-f'(a)}{g'(b)-g'(a)} - \frac{f''(b)}{g''(b)} > 0.$$

Similarly, we can obtain that $\frac{\partial \Phi}{\partial a} < 0$, because this is the same as to show that

$$(28) \quad \frac{f'(b) - f'(a)}{g'(b) - g'(a)} - \frac{f''(a)}{g''(a)} > 0.$$

Thus, by the implicit function theorem there exists C^1 function $a(s)$ in some neighborhood of b_1 such that $a'(s) = -\frac{\Phi'_b}{\Phi'_a} < 0$, and the pair $(a(s), s)$ solves (22).

Now we want to explain that the function $a(s)$ can be defined on $(c, b_1]$, and, moreover, $\lim_{s \rightarrow c+0} a(s) = c$. Indeed, whenever $a(s) \in (a_1, c)$ and $s \in (c, b_1)$ we can use implicit function theorem, and we can extend the function $a(s)$. It is clear that for each s we have $a(s) \in [a_1, c)$ and $s \in (c, b_1)$. Indeed, if $a(s), s \in (a_1, c]$, or $a(s), s \in [c, b_1]$ then (22) has a definite sign (see (24)). It follows that $\alpha(s) \in C^1((c, b_1])$, and condition $a'(s) < 0$ implies $\lim_{s \rightarrow c+0} a(s) = c$. Hence $a(s) \in C([c, b_1])$. \square

It is worth mentioning that we didn't use the fact that torsion of $(s, g(s), f(s))$ changes sign from $+$ to $-$. The only thing we needed was that torsion changes sign.

Let a_1 and b_1 be any solutions of equation (22) from Lemma 8, and let $a(s)$ be any function from Lemma 9. Fix an arbitrary $s_1 \in (c, b_1)$ and consider a foliation $\Theta_{\text{cup}}([s_1, b_1], g)$ constructed by $a(s)$ (see Lemma 7). Let \mathbf{B} be a function defined by (15) where

$$(29) \quad t_2(s_1) = \frac{f'(s_1) - f'(a(s_1))}{g'(s_1) - g'(a(s_1))}.$$

Set $\Omega_{\text{cup}} = \Omega(\Theta_{\text{cup}}([s_1, b_1], g))$, and let $\overline{\Omega_{\text{cup}}}$ be the closer of Ω_{cup} .

Lemma 10. *The function \mathbf{B} satisfies the following properties*

1. $\mathbf{B} \in C^2(\Omega_{\text{cup}}) \cap C^1(\overline{\Omega_{\text{cup}}})$.
2. $\mathbf{B}(a(s), g(a(s))) = f(a(s))$ for all $s \in [s_1, b_1]$.
3. \mathbf{B} is a concave function in $\overline{\Omega_{\text{cup}}}$.

Proof. The first property follows from Lemma 3 and the fact that $\nabla B(x) = t(s)$ for $s = s(x)$, where $s(x)$ is a continuous function in $\overline{\Omega_{\text{cup}}}$.

We are going to check the second property. We recall (see (13)) that $t_1(s) = f'(s) - t_2(s)g'(s)$. Condition (29) implies that

$$(30) \quad t_1(s_1) + t_2(s_1)g'(a(s_1)) = f'(a(s_1)).$$

Let $\mathbf{B}(a(s), g(a(s))) = \tilde{f}(a(s))$. After differentiation of this equality we get $t_1(s_1) + t_2(s_1)g'(a(s_1)) = \tilde{f}'(a(s_1))$. Hence, (30) implies that $f'(a(s_1)) = \tilde{f}'(a(s_1))$. It is clear that

$$\begin{aligned} t_1(s) + t_2(s)g'(s) &= f'(s), \\ t_1(s) + t_2(s)g'(a(s)) &= \tilde{f}'(a(s)), \\ t_1(s)(s - a(s)) + t_2(s)(g(s) - g(a(s))) &= f(s) - \tilde{f}(a(s)), \end{aligned}$$

which implies

$$\begin{vmatrix} 1 & 1 & s-a(s) \\ g'(s) & g'(a(s)) & g(s)-g(a(s)) \\ f'(s) & \tilde{f}'(a(s)) & f(s)-\tilde{f}(a(s)) \end{vmatrix} = 0.$$

This equality can be rewritten as following

$$f' \cdot \begin{vmatrix} 1 & s-a(s) \\ g'(a(s)) & g(s)-g(a(s)) \end{vmatrix} - \tilde{f}'(a) \begin{vmatrix} 1 & s-a(s) \\ g' & g(s)-g(a(s)) \end{vmatrix} + (f - \tilde{f}(a))(g'(a(s)) - g'(s)) = 0.$$

By virtue of Lemma 9 we have the same equality as above except \tilde{f} is replaced by f . We subtract one from another one

$$[f(a(s)) - \tilde{f}(a(s))] + [f'(a(s)) - \tilde{f}'(a(s))] \cdot \frac{\begin{vmatrix} 1 & s-a(s) \\ g' & g(s)-g(a(s)) \end{vmatrix}}{g'(a(s)) - g'(s)} = 0.$$

Note that

$$\frac{\begin{vmatrix} 1 & s-a(s) \\ g' & g(s)-g(a(s)) \end{vmatrix}}{g'(a(s)) - g'(s)} < 0$$

and $a(s)$ is invertible. Therefore we get the following type of differential equation $z(u)B(u) + z'(u) = 0$ where $B \in C^1([a(b_1), a(s_1)])$, $z(u) = f(u) - \tilde{f}(u)$ and $B < 0$. Condition $z'(a(s_1)) = 0$ implies $z(a(s_1)) = 0$. Note that $z = 0$ is a trivial solution. Therefore, by uniqueness of ODE we get $z = 0$.

We are going to check concavity of **B**. Let \mathcal{F} be a force function corresponding to **B**. By Corollary 2 we only need to check that $\mathcal{F}(s_1) \leq 0$. Note that (18) and (29) imply

$$\mathcal{F}(s_1) = \frac{f''(s_1)}{g''(s_1)} - t_2(s_1) = \frac{f''(s_1)}{g''(s_1)} - \frac{f'(s_1) - f'(a(s_1))}{g'(s_1) - g'(a(s_1))}$$

which is negative by (27). \square

Remark 9. The above lemma is true for all choices $s_1 \in (c, b_1)$. If we send s_1 to c then one can easily see that $\lim_{s_1 \rightarrow c+} t_2(s_1) = 0$, therefore the force function \mathcal{F} takes the following form

$$\mathcal{F}(s) = \int_c^s \left[\frac{f''(y)}{g''(y)} \right]' \exp \left(- \int_y^s \frac{g''(r)}{K(r)} \cos \theta(r) dr \right) dy.$$

This is another way to show that the force function is nonpositive.

The next lemma shows that the regardless of the choices of initial solution (a_1, b_1) of (22), the constructed function $a(s)$ by Lemma 9 is unique (i.e. it does not depend on the pair (a_1, b_1)).

Lemma 11. Let pairs (a_1, b_1) , $(\tilde{a}_1, \tilde{b}_1)$ solve (22), and let $a(s), \tilde{a}(s)$ be a corresponding functions obtained by Lemma 9. Then $a(s) = \tilde{a}(s)$ on $[c, \min\{b_1, \tilde{b}_1\}]$.

Proof. By uniqueness of implicit function theorem we only need to show existence of $s_1 \in (c, \min\{b_1, \tilde{b}_1\})$ such that $a(s_1) = \tilde{a}(s_1)$. Without loss of generality assume that $\tilde{b}_1 = b_1 = s_2$. We can also assume that $\tilde{a}(s_2) > a(s_2)$, because other cases can be solved in a similar way. Let $\Theta = \Theta_{\text{cup}}([c, s_2], g)$ and $\tilde{\Theta} = \tilde{\Theta}_{\text{cup}}([c, s_2], g)$ be foliations corresponding to the functions $a(s)$ and $\tilde{a}(s)$. Let \mathbf{B} and $\tilde{\mathbf{B}}$ be functions corresponding to these foliations from Lemma 10. We consider a chord T in \mathbb{R}^3 joining the following points $(a(s_1), g(a(s_1)), f(a(s_1)))$ and $(s_1, g(s_1), f(s_1))$ (see Figure 7). We want to show that the chord T belongs to the graph of $\tilde{\mathbf{B}}$. Indeed, concavity of $\tilde{\mathbf{B}}$ (see Lemma 10) implies that the chord T lies below the graph of $\tilde{\mathbf{B}}(x_1, x_2)$, where $(x_1, x_2) \in \Omega(\tilde{\Theta})$. Moreover, concavity of \mathbf{B} , $\Omega(\tilde{\Theta}) \subset \Omega(\Theta)$ and the fact that the graph $\tilde{\mathbf{B}}$ consists by chords joining the points of the curve $(t, g(t), f(t))$ imply that the graph \mathbf{B} lies above the graph $\tilde{\mathbf{B}}$. In particular the chord T , belonging to the graph \mathbf{B} lies above the graph $\tilde{\mathbf{B}}$. It can happen if and only if when T belongs to the graph $\tilde{\mathbf{B}}$.

Now we show that the if $s_1 < s_2$, then the torsion of the curve $(s, g(s), f(s))$ is zero for $s \in [s_1, s_2]$. Indeed, let \tilde{T} be a chord in \mathbb{R}^3 which joins the points $(a(s_1), g(a(s_1)), f(a(s_1)))$ and $(s_2, g(s_2), f(s_2))$. We consider the tangent plane $L(x)$ to the graph $\tilde{\mathbf{B}}$ at point $(x_1, x_2) = (a(s_1), g(a(s_1)))$. This tangent plane must contain both chords T and \tilde{T} , and it must be a tangent at these chords to the surface. Concavity of $\tilde{\mathbf{B}}$ implies that the tangent plane L coincides with $\tilde{\mathbf{B}}$ at points belonging to the triangle, which is convex hull of the points $(a(s_1), g(a(s_1)))$, $(s_1, g(s_1))$ and $(s_2, g(s_2))$. Therefore, it is clear that the tangent plane L coincides with $\tilde{\mathbf{B}}$ on the segments $\ell \in \tilde{\Theta}$ with the endpoint at $(s, g(s))$ for $s \in [s_1, s_2]$. Thus $L((s, g(s))) = \tilde{\mathbf{B}}((s, g(s)))$ for any $s \in [s_1, s_2]$. This means that the torsion of the curve $(s, g(s), f(s))$ is zero on $s \in [s_1, s_2]$ which contradicts to our assumption about torsion. Therefore $s_1 = s_2$. \square

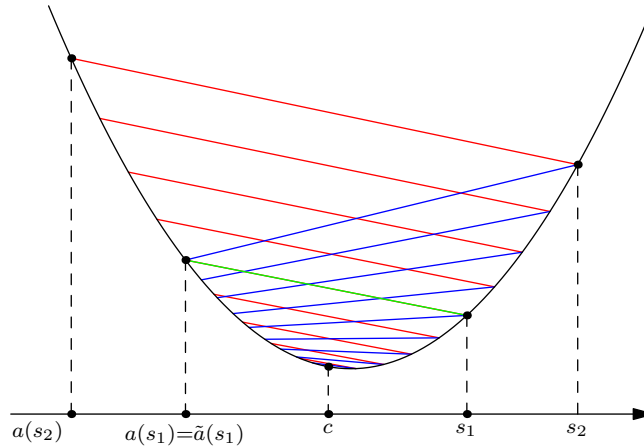


FIGURE 7. Uniqueness of cup

Corollary 4. *In the conditions of Lemma 8 for all $0 < P < \min\{c - a_0, b_0 - c\}$ there exists unique pair (a_1, b_1) which solves (22) such that $b_1 - a_1 = P$.*

The above corollary implies that if the pairs (a_1, b_1) and $(\tilde{a}_1, \tilde{b}_1)$ solve (22), then $a_1 \neq \tilde{a}_1$ and $b_1 \neq \tilde{b}_1$, and one of the following conditions holds: $(a_1, b_1) \subset (\tilde{a}_1, \tilde{b}_1)$, or $(\tilde{a}_1, \tilde{b}_1) \subset (a_1, b_1)$.

Remark 10. *The function $a(s)$ is defined on the right of the point c . We extend naturally its definition on the left of the interval by the following way: $a(s) \stackrel{\text{def}}{=} a^{-1}(s)$.*

4. CONSTRUCTION OF THE BELLMAN FUNCTION

4.1. Reduction to the two dimensional case. We are going to construct the Bellman function for the case $p < 2$. The case $p = 2$ is trivial, and the case $p > 2$ was solved in [4]. We should mention that the reduction presented in this subsection is similar to those presented in [5],[4].

From the definition of H it follows that

$$(31) \quad H(x_1, x_2, x_3) = H(|x_1|, |x_2|, x_3) \quad \text{for all } (x_1, x_2, x_3) \in \Omega.$$

Also note the homogeneity condition

$$(32) \quad H(\lambda x_1, \lambda x_2, \lambda^p x_3) = \lambda^p H(x_1, x_2, x_3) \quad \text{for all } \lambda \geq 0.$$

These two conditions (31), (32), which follow from the nature of the boundary data $(x^2 + \tau^2 y^2)^{2/p}$, make the construction of H easier. However, in order to construct the function H , this information is not necessary. Further, we assume that H is $C^1(\Omega)$ smooth. Then from the symmetry (31) it follows that

$$(33) \quad \frac{\partial H}{\partial x_j} = 0 \quad \text{on } x_j = 0 \quad \text{for } j = 1, 2.$$

For convenience, as in [4], we rotate the system of coordinates (x_1, x_2, x_3) . Namely, let

$$(34) \quad y_1 \stackrel{\text{def}}{=} \frac{x_1 + x_2}{2}, \quad y_2 \stackrel{\text{def}}{=} \frac{x_2 - x_1}{2}, \quad y_3 \stackrel{\text{def}}{=} x_3.$$

We define

$$N(y_1, y_2, y_3) \stackrel{\text{def}}{=} H(y_1 - y_2, y_1 + y_2, y_3) \quad \text{on } \Omega_1,$$

where $\Omega_1 = \{(y_1, y_2, y_3) : y_3 \geq 0, |y_1 - y_2|^p \leq y_3\}$. It is clear that for fixed y_1 , the function N is concave with respect to variables y_2, y_3 , moreover, for fixed y_2 the function N is concave with respect to the rest of variables. The symmetry (31) for N turns into the following condition

$$(35) \quad N(y_1, y_2, y_3) = N(y_2, y_1, y_3) = N(-y_1, -y_2, y_3).$$

Thus the function N is sufficient to construct on the domain

$$\Omega_2 \stackrel{\text{def}}{=} \{(y_1, y_2, y_3) : y_1 \geq 0, -y_1 \leq y_2 \leq y_1, (y_1 - y_2)^p \leq y_3\}.$$

So the condition (33) turns into

$$(36) \quad \frac{\partial N}{\partial y_1} = \frac{\partial N}{\partial y_2} \quad \text{on the hyperplane } y_2 = y_1,$$

$$(37) \quad \frac{\partial N}{\partial y_1} = -\frac{\partial N}{\partial y_2} \quad \text{on the hyperplane } y_2 = -y_1.$$

The boundary condition (7) becomes:

$$(38) \quad N(y_1, y_2, |y_1 - y_2|^p) = ((y_1 + y_2)^2 + \tau^2(y_1 - y_2)^2)^{p/2}.$$

Homogeneity condition (32) implies that $N(\lambda y_1, \lambda y_2, \lambda^p y_3) = \lambda^p N(y_1, y_2, y_3)$ for $\lambda \geq 0$. We choose $\lambda = 1/y_1$, and we obtain that

$$(39) \quad N(y_1, y_2, y_3) = y_1^p N(1, \frac{y_2}{y_1}, \frac{y_3}{y_1^p})$$

Suppose we are able to construct the function $M(y_2, y_3) \stackrel{\text{def}}{=} N(1, y_2, y_3)$ on

$$\Omega_3 \stackrel{\text{def}}{=} \{(y_2, y_3) : -1 \leq y_2 \leq 1, (1 - y_2)^p \leq y_3\}$$

with the following conditions:

1. M is concave in Ω_3
2. M satisfies (38) for $y_1 = 1$.
3. Extension of M onto Ω_1 via formulas (39) and (35) is a function with the properties of N (see (36), (37), and concavity of N).
4. M is minimal among those who satisfy the conditions 1,2,3.

Then this extended function M should be N . So we are going to construct M on Ω_3 . We denote

$$(40) \quad g(t) \stackrel{\text{def}}{=} (1 - t)^p, \quad t \in [-1, 1],$$

$$(41) \quad f(t) \stackrel{\text{def}}{=} ((1 + t)^2 + \tau^2(1 - t)^2)^{p/2}, \quad t \in [-1, 1].$$

Then we have the boundary condition

$$(42) \quad M(t, g(t)) = f(t), \quad t \in [-1, 1].$$

We differentiate the condition (39) by y_1 at point $(y_1, y_2, y_3) = (1, -1, y_3)$ and obtain that

$$\frac{\partial N}{\partial y_1}(1, -1, y_3) = pN(1, -1, y_3) + \frac{\partial N}{\partial y_2}(1, -1, y_3) - py_3 \frac{\partial N}{\partial y_3}, \quad y_3 \geq 0.$$

Now we use (37), so we obtain another requirement for $M(y_2, y_3)$:

$$(43) \quad 0 = pM(-1, y_3) + 2 \frac{\partial M}{\partial y_2}(-1, y_3) - py_3 \frac{\partial M}{\partial y_3}(-1, y_3), \quad \text{for } y_3 \geq 0.$$

Similarly, we differentiate (39) by y_1 at point $(y_1, y_2, y_3) = (1, 1, y_3)$ and use the (36), so we obtain

$$(44) \quad 0 = pM(1, y_3) - 2 \frac{\partial M}{\partial y_2}(1, y_3) - py_3 \frac{\partial M}{\partial y_3}(1, y_3), \quad \text{for } y_3 \geq 0.$$

So in order to satisfy conditions (36) and (37), the requirements (43) and (44) are necessary. The reader can note easily that these requirements are also sufficient to satisfy these conditions. This can be verified directly, so we left it to the reader. The minimum between two concave functions with fixed boundary data is concave function with the same boundary data. Note also that the conditions (43) and (44) still fulfilled after taking the minimum. Thus it is quite reasonable to construct a candidate for $M(y_2, y_3)$ as a minimal concave function on Ω_3 with boundary conditions (42), (43) and (44). We remind that we should also have the concavity of the extended function $N(y_1, y_2, y_3)$ with respect to variables y_1, y_3 for each fixed y_2 . This condition can be verified after the construction of the function $M(y_2, y_3)$.

4.2. Construction of a candidate for M. We are going to construct candidate **B** for M . Firstly, we show that for $\tau > 0$, the torsion σ_γ of the boundary curve $\gamma(t) \stackrel{\text{def}}{=} (t, g(t), f(t))$ on $t \in (-1, 1)$, where f, g are defined by (40) and (41), changes sign once from $+$ to $-$. We call this point root of a cup. We construct a cup around this point. Note that $g' < 0, g'' > 0$ on $[-1, 1)$. Therefore

$$\text{sign } \tau_\gamma = \text{sign} \left(f''' - \frac{g'''}{g''} f'' \right) = \text{sign} \left(f''' - \frac{2-p}{1-t} f'' \right) = \text{sign}(v(t)),$$

where

$$v(t) \stackrel{\text{def}}{=} -(1+\tau^2)^2(p-1)t^3 + (1+\tau^2)(3\tau^2 + \tau^2 p + 3 - 3p)t^2 + (2\tau^2 p - 9\tau^4 + \tau^4 p + 3 - 3p - 6\tau^2)t - p + 5\tau^4 + 2\tau^2 p - \tau^4 p - 10\tau^2 + 1.$$

Note that $v(-1) = 16\tau^4 > 0$ and $v(1) = -8((p-1) + \tau^2) < 0$. So the function $v(t)$ at least one times change the sign from $+$ to $-$. Now, we show that $v(t)$ has only one root. For $\tau^2 < \frac{3(p-1)}{3-p}$, note that the linear function $v''(t)$ is nonnegative i.e. $v''(-1) = 8\tau^2 p(1 + \tau^2) > 0$, $v''(1) = -4(1 + \tau^2)(\tau^2 p - 3\tau^2 + 3p - 3) \geq 0$. Therefore, the convexity of $v(t)$ implies the uniqueness of the root $v(t)$ on $[-1, 1]$.

Suppose $\tau^2 < \frac{3(p-1)}{3-p}$, then we will show that $v' \leq 0$ on $[-1, 1]$. Indeed, the discriminant of the quadratic function $v'(x)$ has the expression

$$D = 16\tau^2(\tau^2 + 1)^2((3-p)^2\tau^2 - 9(p-1)).$$

which is negative for $0 < \tau^2 < \frac{3(p-1)}{3-p}$. Moreover, $v'(-1) = -4\tau^2(\tau^2 p + 3\tau^2 + 3) < 0$. Thus we obtain that v' is negative.

We denote the root of v by c . It is an appropriate time to make the following remark

Remark 11. Note that $v(-1 + 2/p) < 0$. Indeed,

$$v(-1 + 2/p) = \frac{(3p-2)(p^2 - 2p - 4)\tau^4 + (16 + 5p^3 - 8p^2 - 16p)\tau^2 + 8(1-p)}{p^3}$$

which is negative because coefficients of τ^4, τ^2, τ^0 are negative. Therefore, this inequality implies that $c < -1 + 2/p$.

Consider $a = -1$ and $b = 1$ then the left side of (22) takes the positive value $-2^{2p-1}p(1-p)$. However, if we consider $a = -1$ and $b = c$, then the proof of Lemma 8 (see 24) implies that the left side of (22) is negative. Therefore, there exists unique $s_0 \in (c, 1)$ such that the pair $(-1, s_0)$ solves (22). Uniqueness follows from Corollary 4. Equation (22) for the pair $(-1, s_0)$ is equivalent to the following equation $u\left(\frac{1+s_0}{1-s_0}\right) = 0$, where

$$(45) \quad u(z) \stackrel{\text{def}}{=} \tau^p(p-1)(\tau^2 + z^2)^{(2-p)/2} - \tau^2(p-1) + (1+z)^{2-p} - z(2-p) - 1.$$

Lemma 9 gives the function $a(s)$, and Lemma 10 gives concave function $\mathbf{B}(y_2, y_3)$ for $s_1 = c$ with the foliation $\Theta_{\text{cup}}((c, s_0], g)$ in the domain $\Omega(\Theta_{\text{cup}}((c, s_0], g))$.

The above explanation implies the following: pick any point $\tilde{y}_2 \in (-1, 1)$.

Corollary 5. *The following inequalities $s_0 < \tilde{y}_2$, $s_0 = \tilde{y}_2$ and $\tilde{y}_2 > s_0$ are equivalent to the following inequalities respectively: $u\left(\frac{1+\tilde{y}_2}{1-\tilde{y}_2}\right) < 0$, $u\left(\frac{1+\tilde{y}_2}{1-\tilde{y}_2}\right) = 0$ and $u\left(\frac{1+\tilde{y}_2}{1-\tilde{y}_2}\right) > 0$.*

Now we are going to extend C^1 smoothly the function \mathbf{B} on the upper part of the cup. Recall that we are looking for a minimal concave function. If we construct a function with a foliation $\Theta([s_0, \tilde{y}_2], g)$ where $\tilde{y}_2 \in (s_0, 1)$ then the best thing we can do according to Lemma 6 and Lemma 5 is to minimize $\sin(\theta_{\text{cup}}(s_0) - \theta(s_0))$ where $\theta_{\text{cup}}(s)$ is an argument function of $\Theta_{\text{cup}}((c, s_0], g)$ and $\theta(s)$ is an argument function of $\Theta([s_0, \tilde{y}_2], g)$. In other words we need to choose the segments from $\Theta([s_0, \tilde{y}_2], g)$ close enough to the segments of $\Theta_{\text{cup}}((c, s_0], g)$.

Thus, we are going to try to construct set of segments $\Theta([s_0, \tilde{y}_2])$ so that they start from $(s, g(s), f(s))$, $s \in [s_0, \tilde{y}_2]$, and they go to the boundary $y_2 = -1$ of Ω_3 .

We explain how the conditions (43) and (44) allow us to construct such type of foliation $\Theta([s_0, \tilde{y}_2], g)$ in unique way. Let $\ell(y)$ be a segment with the endpoints $(s, g(s))$ where $s \in (s_0, \tilde{y}_2)$ and $(-1, h(s))$ (see Figure 8).

Let $t(s) = (t_1(s), t_2(s)) = \nabla \mathbf{B}(y)$ where $s = s(y)$ be a corresponding gradient function. Then (43) takes the form

$$(46) \quad 0 = p\mathbf{B}(-1, h(s)) + 2t_1(s) - ph(s)t_2(s).$$

We differentiate this expression with respect to s , and we obtain

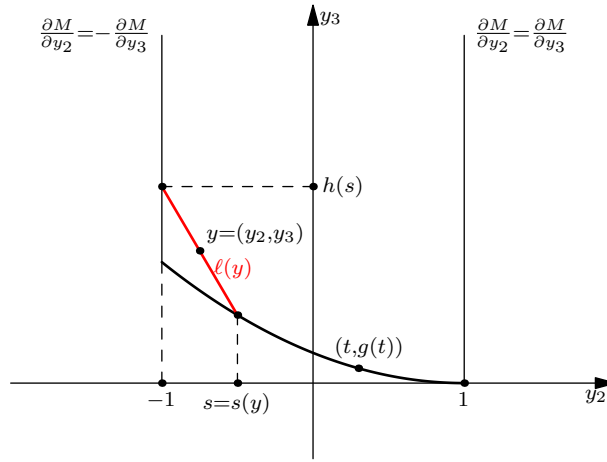
$$(47) \quad 2t'_1(s) - ph(s)t'_2(s) = 0.$$

Then according to (12) we find the function $\tan \theta(s)$, and, hence, we find the quantity $h(s)$

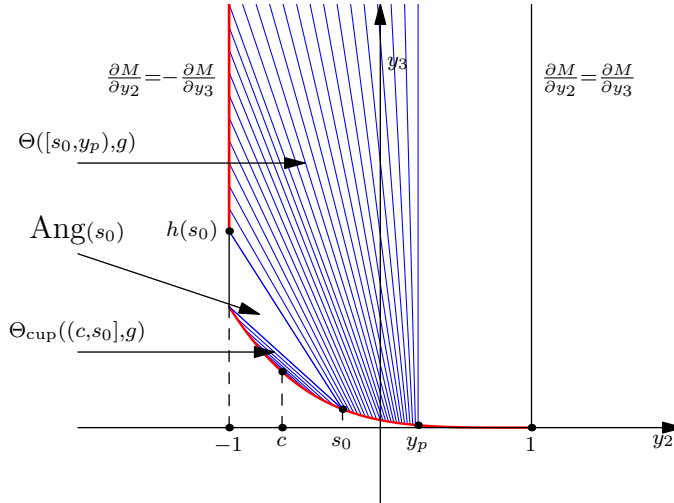
$$\tan \theta(s) = -\frac{ph(s)}{2} \Leftrightarrow \frac{h(s) - g(s)}{s + 1} = \frac{ph(s)}{2},$$

therefore,

$$(48) \quad h(s) = \frac{2g(s)}{p} \left(\frac{1}{y_p - s} \right) \quad \text{where} \quad y_p \stackrel{\text{def}}{=} -1 + \frac{2}{p}.$$

FIGURE 8. Segment $\ell(y)$

We see that the function $h(s)$ is well defined, it increases, and it is differentiable on $-1 \leq s < y_p$. So we conclude that if $s_0 < y_p$ then we are able to construct the set of segments $\Theta([s_0, y_p], g)$ that pass through the points $(s, g(s))$, where $s \in [s_0, y_p]$ and through the boundary $y_2 = -1$ (see Figure 9).

FIGURE 9. Foliations $\Theta_{\text{cup}}((c, s_0], g)$ and $\Theta([s_0, y_p], g)$

It is easy to check that $\Theta([s_0, y_p], g)$ is a foliation. So choosing value $t_2(s_0)$ of **B** on $\Omega(\Theta([s_0, y_p], g))$ according to Lemma 6, then by Corollary 3 we have constructed concave function **B** in the domain $\Omega(\Theta_{\text{cup}}((c, s_0], g)) \cup \text{Ang}(s_0) \cup \Omega(\Theta([s_0, y_p], g))$.

It is clear that the foliation $\Theta([s_0, y_p], g)$ exists as long as $s_0 < y_p$. Note that $\frac{1+y_p}{1-y_p} = \frac{1}{p-1}$. Therefore Corollary 5 implies the following remark

Remark 12. The following inequalities $s_0 < y_p$, $s_0 = y_p$ and $s_0 > y_p$ are equivalent to the following inequalities respectively: $u\left(\frac{1}{p-1}\right) < 0$, $u\left(\frac{1}{p-1}\right) = 0$ and $u\left(\frac{1}{p-1}\right) > 0$.

At point y_p segments from $\Theta([s_0, y_p], g)$ become vertical. After the point $(y_p, g(y_p))$ we should consider vertical segments $\Theta([y_p, 1], g)$ (see Figure 10), because by

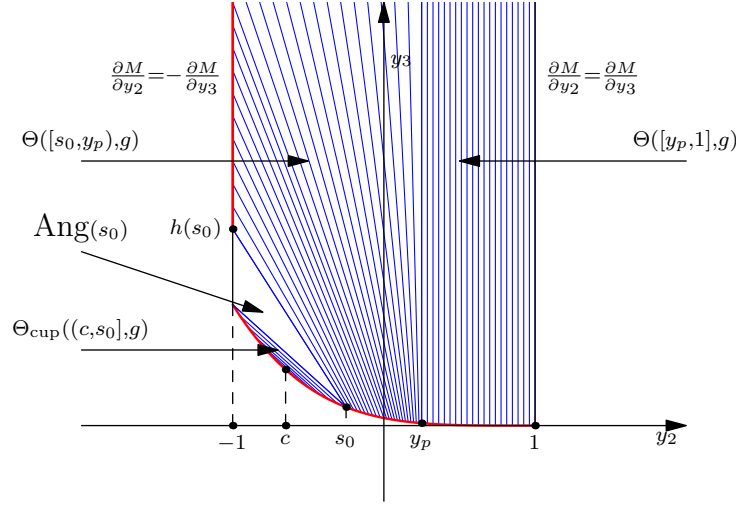


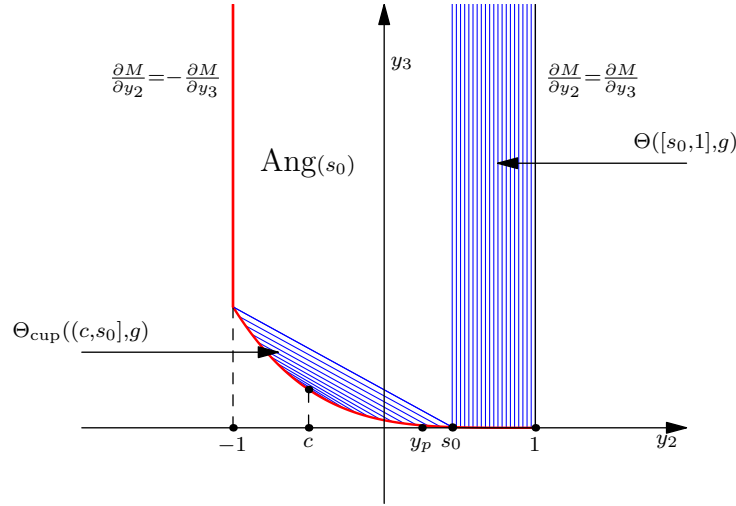
FIGURE 10. Case $u\left(\frac{1}{p-1}\right) < 0$

Lemma 5 this corresponds to a minimal function. Surely $\Theta([y_p, 1], g)$ is a foliation. Again, choosing value $t_2(y_p)$ of \mathbf{B} on $\Omega(\Theta([y_p, 1], g))$ according to Lemma 6, then by Corollary 3 we have constructed concave function \mathbf{B} on Ω_3 . Note that if $s_0 \geq y_p$, which corresponds to the inequality $u\left(\frac{1}{p-1}\right) > 0$, then we don't have a foliation $\Theta([s_0, y_p], g)$. In this case we consider only vertical segments $\Theta([s_0, 1], g)$ (see Figure 11), and again, choosing value $t_2(s_0)$ of \mathbf{B} on $\Omega(\Theta([s_0, 1], g))$ according to Lemma 6, then by Corollary 3 we construct a concave function \mathbf{B} on Ω_3 . We believe that $\mathbf{B} = M$.

We still have to check the requirements (43) and (44). The crucial role plays symmetry of the boundary data of N . Further given proofs work for both of the cases $y_p < s_0$ and $y_p \geq s_0$. Therefore we don't consider them separately.

The requirement (44) follows immediately. Indeed, the condition (15) at point $y = (1, y_3)$ (note that in (15) instead of $x = (x_1, x_2)$ we consider $y = (y_2, y_3)$) implies that $\mathbf{B}(1, y_3) = f(1) + t_2(1)(y_3 - g(1))$. Therefore, the requirement (44) takes the form $0 = pf(1) - 2t_1(1)$. Using (13), we obtain that $t_1(1) = f'(1)$. Therefore, we see that $pf(1) - 2t_1(1) = pf(1) - 2f'(1) = 0$.

Now, we are going to obtain requirement (43) which is the same as (46). The quantities t_1, t_2 of \mathbf{B} with the foliation $\Theta([s_0, y_p], g)$ satisfy the condition (47),

FIGURE 11. Case $u\left(\frac{1}{p-1}\right) \geq 0$

which was obtained by differentiation of (46). So we only need to check the condition (46) at initial point $s = s_0$. If we substitute the expression of \mathbf{B} from (15) into (46), then (46) turns into the following equivalent condition.

$$(49) \quad t_1(s)(s - y_p) + t_2(s)g(s) = f(s).$$

Note that (13) allows us the rewrite (49) into the following equivalent condition

$$(50) \quad t_2(s) = \frac{f(s) - (s - y_p)f'(s)}{g(s) - (s - y_p)g'(s)}.$$

And as it was mentioned above we only need to check condition (50) at point $s = s_0$ i.e.

$$(51) \quad t_2(s_0) = \frac{f(s_0) - (s_0 - y_p)f'(s_0)}{g(s_0) - (s_0 - y_p)g'(s_0)}.$$

On the other hand, if we differentiate the boundary condition $\mathbf{B}(s, g(s)) = f(s)$ at points $s = s_0, -1$, then we obtain

$$\begin{aligned} t_1(s_0) + t_2(s_0)g'(-1) &= f'(-1), \\ t_1(s_0) + t_2(s_0)g'(s_0) &= f'(s_0). \end{aligned}$$

Thus we can find actual value of $t_2(s_0)$ which is

$$(52) \quad t_2(s_0) = \frac{f'(-1) - f'(s_0)}{g'(-1) - g'(s_0)}.$$

So this two values (52) and (51) must coincide. In other words we need to show

$$(53) \quad \frac{f(s_0) - (s_0 - y_p)f'(s_0)}{g(s_0) - (s_0 - y_p)g'(s_0)} = \frac{f'(-1) - f'(s_0)}{g'(-1) - g'(s_0)}.$$

It will be convenient for us to work with the following notations up to the rest of the current subsection. We denote $g(-1) = g_-$, $g'(-1) = g'_-$, $f(-1) = f_-$, $f'(-1) = f'_-$, $g(s_0) = g$, $g'(s_0) = g'$, $f(s_0) = f$, $f'(s_0) = f'$. The condition (53) is equivalent to

$$(54) \quad \begin{aligned} s_0 &= \frac{fg'_- + f'g - fg' - gf'_-}{f'g'_- - g'f'_-} + y_p = \\ &= \left(\frac{fg'_- + f'g - fg' - gf'_-}{f'g'_- - g'f'_-} - 1 \right) + \frac{2}{p}. \end{aligned}$$

On the other hand, from (22) for the pair $(-1, s_0)$ we obtain that

$$s_0 = \left(\frac{fg'_- + f'g - fg' - gf'_-}{f'g'_- - g'f'_-} - 1 \right) + \frac{f'g_- + g'_-f_- - g'f_- - f'_-g_-}{g'f'_- - f'g'_-}.$$

So, from (54) we see that it suffices to show that

$$\frac{f'g_- + g'_-f_- - g'f_- - f'_-g_-}{g'f'_- - f'g'_-} = \frac{2}{p}.$$

We note that $g'_- = -(p/2)g_-$, $f'_- = -(p/2)f_-$, hence, $g'_-f_- = f'_-g_-$, therefore, we have

$$\frac{f'g_- + g'_-f_- - g'f_- - f'_-g_-}{g'f'_- - f'g'_-} = \frac{f'g_- - g'f_-}{g'f'_- - f'g'_-} = \frac{2}{p}.$$

4.3. Concavity in another direction. We are going to check the concavity of the extended function N via \mathbf{B} in another direction. It is worth mentioning that the both of the cases $y_p < s_0$, $y_p \geq s_0$ do not play any role in the following computations, therefore we consider them together. We define a candidate for N as

$$(55) \quad N(y_1, y_2, y_3) \stackrel{\text{def}}{=} y_1^p \mathbf{B}(1, y_2/y_1, y_3/y_1^p) \quad \text{for} \quad \left(\frac{y_2}{y_1}, \frac{y_3}{y_1^p} \right) \in \Omega_3,$$

and we extend N to the Ω_1 by (35). Then, as it was already discussed, $N \in C^1(\Omega_1)$. We need the following technical lemma:

Lemma 12.

$$N''_{y_1 y_1} N''_{y_3 y_3} - (N''_{y_1 y_3})^2 = -t'_2 s'_{y_3} p(p-1) y_1^{p-2} \left(s t_1 + g t_2 - f + \frac{y_2}{y_1} t_1 \cdot \left(\frac{2}{p} - 1 \right) \right)$$

where $s = s\left(\frac{y_2}{y_1}, \frac{y_3}{y_1^p}\right)$ and $\left(\frac{y_2}{y_1}, \frac{y_3}{y_1^p}\right) \in \text{int}(\Omega_3) \setminus \text{Ang}(s_0)$.

As it was mentioned in Remark 8, the gradient function $t(s)$ is not necessarily differentiable at point s_0 , this is the reason of the requirement $\left(\frac{y_2}{y_1}, \frac{y_3}{y_1^p}\right) \in \text{int}(\Omega_3) \setminus \text{Ang}(s_0)$ in the lemma. However, from the proof of lemma, the reader can easily see that $N''_{y_1 y_1} N''_{y_3 y_3} - (N''_{y_1 y_3})^2 = 0$ whenever the points $\left(\frac{y_2}{y_1}, \frac{y_3}{y_1^p}\right)$ belong to the interior of the domain $\text{Ang}(s_0)$.

Proof. Definition of the candidate N (see (55)) implies $N''_{y_3 y_3} = t'_2(s) s'_{y_3}$, $N''_{y_3 y_1} = t'_2 s'_{y_1}$,

$$(56) \quad N'_{y_1} = y_1^{p-1} \left(p \mathbf{B} \left(\frac{y_2}{y_1}, \frac{y_3}{y_1^p} \right) - t_1 \frac{y_2}{y_1} - p t_2 \frac{y_3}{y_1^p} \right).$$

Condition (15) implies

$$\mathbf{B} \left(\frac{y_2}{y_1}, \frac{y_3}{y_1^p} \right) = f(s) + t_1 \cdot \left(\frac{y_2}{y_1} - s \right) + t_2 \cdot \left(\frac{y_3}{y_1^p} - g(s) \right).$$

We substitute this expression of $\mathbf{B} \left(\frac{y_2}{y_1}, \frac{y_3}{y_1^p} \right)$ into (56), and we obtain:

$$(57) \quad N'_{y_1} = y_1^{p-1} \left(p f + \frac{y_2}{y_1} t_1 (p-1) - p s t_1 - p g t_2 \right).$$

Condition $\left(\frac{y_2}{y_1}, \frac{y_3}{y_1^p} \right) \in \text{int}(\Omega_3) \setminus \text{Ang}(s_0)$ implies the equality $N''_{y_1 y_3} = N''_{y_3 y_1}$ which in turn gives

$$t'_2 s'_{y_1} = y_1^{p-1} \left(p f' + \frac{y_2}{y_1} t'_1 (p-1) - (p s t_1 + p g t_2)'_s \right) s'_{y_3}.$$

Hence

$$(58) \quad t'_2 \cdot (s'_{y_1})^2 = y_1^{p-1} \left(p f' + \frac{y_2}{y_1} t'_1 (p-1) - (p s t_1 + p g t_2)'_s \right) s'_{y_3} s'_{y_1}.$$

We keep in mind this identity and continue our calculations

$$\begin{aligned} N''_{y_1 y_1} &= (p-1) y_1^{p-2} \left(p f + \frac{y_2}{y_1} t_1 (p-2) - p s t_1 - p g t_2 \right) + \\ & y_1^{p-1} \left(p f' + \frac{y_2}{y_1} t'_1 (p-1) - (p s t_1 + p g t_2)'_s \right) s'_{y_1}. \end{aligned}$$

So, finally we obtain

$$N''_{y_1 y_1} N''_{y_3 y_3} - (N''_{y_1 y_3})^2 = t'_2 (N''_{y_1 y_1} s'_{y_3} - t'_2 (s'_{y_1})^2).$$

Now we use the identity (58), and we substitute the expression $t'_2 (s'_{y_1})^2$:

$$\begin{aligned} & N''_{y_1 y_1} N''_{y_3 y_3} - (N''_{y_1 y_3})^2 = \\ & t'_2 s'_{y_3} \left(N''_{y_1 y_1} - y_1^{p-1} \left(p f' + \frac{y_2}{y_1} t'_1 (p-1) - (p s t_1 + p g t_2)'_s \right) s'_{y_1} \right) = \\ & t'_2 s'_{y_3} \left((p-1) y_1^{p-2} \left(p f + \frac{y_2}{y_1} t_1 (p-2) - p s t_1 - p g t_2 \right) \right) = \\ & - t'_2 s'_{y_3} p (p-1) y_1^{p-2} \left(s t_1 + g t_2 - f + \frac{y_2}{y_1} t_1 \cdot \left(\frac{2}{p} - 1 \right) \right). \end{aligned}$$

□

Now we are going to consider several cases when the points $(y_2/y_1, y_3/y_1^p)$ belong to different subdomains in Ω_3 . Note that we always have $N''_{y_3y_3} \leq 0$, because of the fact that \mathbf{B} is concave in Ω_3 and (55). So we only have to check that the determinant of the Hessian N is negative. If the determinant of the Hessian is zero, then it is sufficient to ensure that $N''_{y_3y_3}$ is strictly negative, and if $N''_{y_3y_3}$ is also zero, then we need to ensure that $N''_{y_1y_1}$ is nonpositive.

Domain $\Omega(\Theta[s_0, y_p])$.

In this case we can use the equality (49), and we obtain that

$$st_1 + gt_2 - f = y_p t_1.$$

Therefore

$$N''_{y_1y_1} N''_{y_3y_3} - (N''_{y_1y_3})^2 = -t_2' s_{y_3}' p(p-1) y_1^{p-2} t_1 y_p \left(1 + \frac{y_2}{y_1}\right) \geq 0.$$

because $t_1 \geq 0$. Indeed, $t_1(s)$ is continuous on $[c, 1]$, where c is the root of the cup and $\mathbf{B}''_{y_2y_2} = t_1' s_{y_2}' \leq 0$, therefore, because of the fact $s_{y_2}' > 0$, it suffice to check that $t_1(1) \geq 0$ which follows from the following inequality

$$t_1(1) = f'(1) - t_2(1)g'(1) = f'(1) > 0.$$

Domain of linearity $\text{Ang}(s_0)$.

This is the domain, which is obtained by the triangle ABC , where $A = (-1, g(-1))$, $B = (s_0, g(s_0))$, and $C = (-1, h(s_0))$ if $s_0 < y_p$ and by the infinity domain of linearity, which is rectangular type, and which lies between the chords AB , BC' , where $C' = (s_0, +\infty)$ and AC'' , where $C'' = (-1, +\infty)$.

Suppose the points $(y_2/y_1, y_3/y_1^p)$ belong to the interior of $\text{Ang}(s_0)$. Then the gradient function $t(s)$ of \mathbf{B} is constant, and moreover $s\left(\frac{y_2}{y_1}, \frac{y_3}{y_1^p}\right)$ is constant. The fact that determinant of the Hessian is zero in the domain of linearity (note that $s_{y_3}' = 0$), implies that we only need to check $N''_{y_1y_1} < 0$. Equality (57) implies

$$\begin{aligned} N''_{y_1y_1} &= (p-1)y_1^{p-2} \left(pf + \frac{y_2}{y_1} t_1(p-2) - ps_0 t_1 - pgt_2 \right) \leq \\ & (p-1)y_1^{p-2} (pf - ps_0 t_1 - pgt_2 - t_1(p-2)) = 0. \end{aligned}$$

The last equality follows from (49). The above inequality turns into the equality if and only if, when $\frac{y_2}{y_1} = s_0$, this is the boundary point of $\text{Ang}(s_0)$.

Domain of vertical segments.

On vertical segments determinant of the Hessian is zero (for example, because the vertical segment is vertical segment in all directions) and $\mathbf{B}''_{y_3y_3} = 0$, therefore, we must check that $N''_{y_1y_1} \leq 0$. We note that $s(y_2, y_3) = y_2$, therefore,

$$\begin{aligned} N''_{y_1y_1} &= y_1^{p-2} \times [(p-1)(pf + st_1(p-2) - pst_1 - pgt_2) - \\ & - s(pf' - t_1' s - t_1 p - pg't_2)]. \end{aligned}$$

However, from (13) we have $pf' - t_1p - pg't_2 = 0$, therefore,

$$N''_{y_1y_1} = y_1^{p-2} \times [(p-1)(pf - 2st_1 - pgt_2) + s^2t_1'].$$

Condition $t_1' \leq 0$ implies it is sufficient to show that $pf - 2st_1 - pgt_2 \leq 0$. We use (13), and we find $t_1 = f' - g't_2$. Hence,

$$pf - 2st_1 - pgt_2 = pf - gpt_2 - 2s(f' - g't_2) = pf - 2sf' - t_2(gp - 2sg').$$

Note that $gp - 2sg' \geq 0$ (because $s \geq 0$ and $g' \leq 0$), and we recall that from (13) and the fact that on vertical segments t_2 is constant, since we have $\cos \theta(s) = 0$ (see the expression of t_2 from Lemma 2), so t_2 is constant and hence $0 \geq t_1' = f'' - g''t_2$, therefore, we have $t_2 \geq f''/g''$. Therefore,

$$pf - 2sf' - t_2(gp - 2sg') \leq pf - 2sf' - \frac{f''}{g''}(gp - 2sg').$$

Now we recall the values (42), (41), and after direct calculations we obtain that

$$\begin{aligned} pf - 2sf' - \frac{f''}{g''}(gp - 2sg') &= \\ \frac{f(1-s^2)p(p-2)(\tau^2(1+s)^2 + (1-s)^2 + 2\tau^2(1-s^2))}{(p-1)((1+s)^2 + \tau^2(1-s)^2)^2} &\leq 0. \end{aligned}$$

Domain of the cup $\Omega(\Theta_{\text{cup}}((c, s_0], g))$.

The condition that $N''_{y_3y_3}$ is strictly negative in the cup implies that we only need to show $st_2 + gt_3 - f + \frac{y_2}{y_1}t_1(\frac{2}{p} - 1) \geq 0$, where $s = s(y_2/y_1, y_3/y_1^p)$ and the points $y = (y_2/y_1, y_3/y_1^p)$ lie in the cup. We can think that $y_2/y_1 \rightarrow y_2$ and $y_3/y_1^p \rightarrow y_3$ and $s(y_2/y_1, y_3/y_1^p) \rightarrow s(y_2, y_3)$, and we can think that the points (y_2, y_3) lie in the cup. Therefore it suffice to show that $st_2 + gt_3 - f + y_2t_1(\frac{2}{p} - 1) \geq 0$ where $y = (y_2, y_3) \in \Omega(\Theta_{\text{cup}}((c, s_0], g))$. On a segment with the fixed endpoint $(s, g(s))$ the expressions $s, t_1, g(s), t_2, f(s)$ are constant, except of y_2 , so the expression $st_1 + gt_2 - f + y_2t_1(\frac{2}{p} - 1)$ is linear with respect to the y_2 on the each segment of the cup. Therefore, the worst case appears when $y_2 = a(s)$ ($a(s)$ - is the left end (it is abscissa) of the given segment). This is true because $t_1 \geq 0$ (as it was already shown) and $(\frac{2}{p} - 1) \geq 0$. So, as the result, we derive that it is sufficient to prove the inequality

$$(59) \quad st_1 + gt_2 - f + a(s)t_1 \cdot \left(\frac{2}{p} - 1\right) = t_1(s - a(s)) + gt_2 - f + \frac{2a(s)}{p}t_1 \geq 0.$$

We use the identity (15) at the point $y = (a(s), g(a(s)))$, and we find that

$$t_1(s - a(s)) + gt_2 - f = g(a(s))t_2 - f(a(s)).$$

So, we substitute this expression into (59) then we will get that it suffices to prove the inequality:

$$(60) \quad g(a(s))t_2 - f(a(s)) + \frac{2a(s)}{p}t_1 \geq 0.$$

We differentiate condition $\mathbf{B}(a(s), g(a(s))) = f(s)$ with respect to s . Then we find the expression for $t_1(s)$, namely $t_1(s) = f'(a(s)) - t_2(s)g'(a(s))$. After substituting this expression into (60) we obtain that:

$$g(a(s))t_2 - f(a(s)) + \frac{2a(s)}{p}t_1 = \frac{1+z}{g'(z)} \left(\frac{(z-1)(\tau^2+1)f(z)}{((1+z)^2 + \tau^2(1-z)^2)g'(z)} - t_2(s) \right).$$

where $z = a(s)$. So it suffice to show that

$$(61) \quad \frac{(z-1)(\tau^2+1)f(z)}{((1+z)^2 + \tau^2(1-z)^2)g'(z)} - t_2(s) \leq 0$$

because g' is negative. We are going to show that the condition (61) is sufficient to check at point $z = -1$. Indeed, note that $(t_2)'_z \geq 0$ on $[-1, c]$ where c is the root of the cup, and also note that

$$\left(\frac{(z-1)(\tau^2+1)f}{((1+z)^2 + \tau^2(1-z)^2)g'} \right)'_z = \frac{\tau^2+1}{p}(p-2)(1-z)^{-(p-1)}[(1+z)^2 + \tau^2(1-z)^2]^{p/2-2}2(1+z) \leq 0.$$

The condition (61) at point $z = -1$ turns into the following condition

$$t_2(s_0) - \frac{\tau^{p-2}(\tau^2+1)}{p} \geq 0.$$

Now we recall (28) and $t_2(s_0) = (f'(-1) - f'(s_0))/(g'(-1) - g'(s_0))$, therefore we have

$$\begin{aligned} t_2(s_0) - \frac{\tau^{p-2}(\tau^2+1)}{p} &\geq \frac{f''(-1)}{g''(-1)} - \frac{\tau^{p-2}(\tau^2+1)}{p} = \\ &= \frac{\tau^p(p-1)^2 + \tau^{p-2}}{p(p-1)} > 0. \end{aligned}$$

Thus we finish this section by the following remark.

Remark 13. *The only cases remind are the following, when the points $\left(\frac{y_2}{y_1}, \frac{y_3}{y_1^p}\right)$ belong to the boundary of $\text{Ang}(s_0)$ and vertical rays $y_2 = \pm 1$ in Ω_3 . The reader can easily see that in this case concavity of N follows from the observation that $N \in C^1(\Omega_1)$. Symmetry of N covers the rest of the cases when $\left(\frac{y_2}{y_1}, \frac{y_3}{y_1^p}\right) \notin \Omega_3$.*

Thus we have constructed candidate N .

5. SHARP CONSTANTS VIA FOLIATION

5.1. Main theorem. We remind the reader the definition of the functions $u(z)$, $g(s)$, $f(s)$ (see (45), (40), (41)), the value $y_p = -1 + 2/p$ and the definition of the function $a(s)$ (see Lemma 9, Lemma 11 and Remark 10).

Theorem 2. Let $1 < p < 2$, and let G be a martingale transform of F and let $|\mathbb{E}G| \leq \beta|\mathbb{E}F|$. Set $\beta' = \frac{\beta-1}{\beta+1}$.

(i) If $u\left(\frac{1}{p-1}\right) \leq 0$ then

$$(62) \quad \mathbb{E}(\tau^2 F^2 + G^2)^{p/2} \leq \left(\tau^2 + \max \left\{ \beta, \frac{1}{p-1} \right\}^2 \right)^{\frac{p}{2}} \mathbb{E}|F|^p.$$

(ii) If $u\left(\frac{1}{p-1}\right) > 0$ then

$$\mathbb{E}(\tau^2 F^2 + G^2)^{p/2} \leq C(\beta') \mathbb{E}|F|^p.$$

where $C(\beta')$ is continuous nondecreasing, and it is defined by the following way:

$$C(\beta') \stackrel{\text{def}}{=} \begin{cases} (\tau^2 + \beta^2)^{p/2}, & \beta' \geq s^*; \\ \frac{\tau^p}{1 - \frac{2^{2-p}(1-s_0)^{p-1}}{(\tau^2+1)(p-1)(1-s_0)+2(2-p)}}, & \beta' \leq -1 + \frac{2}{p}; \\ \frac{f'(s_1) - f'(a(s_1))}{g'(s_1) - g'(a(s_1))}, & R(s_1, \beta') = 0 \text{ for some } s_1 \in (\beta', s_0); \end{cases}$$

where $s_0 \in (-1 + 2/p, 1)$ is the solution of the equation $u\left(\frac{1+s_0}{1-s_0}\right) = 0$, and the function $R(s, z)$ is defined in the following way

$$R(s, z) \stackrel{\text{def}}{=} -f(s) - \frac{f'(a(s))g'(s) - f'(s)g'(a(s))}{g'(s) - g'(a(s))} (z - s) + \frac{f'(s) - f'(a(s))}{g'(s) - g'(a(s))} g(s) = 0, \quad z \in [-1 + 2/p, s^*], \quad s \in [z, s_0].$$

The value $s^* \in [-1 + 2/p, s_0]$ is the solution of the equation

$$(63) \quad \frac{f'(s^*) - f'(a(s^*))}{g'(s^*) - g'(a(s^*))} = \frac{f(s^*)}{g(s^*)}.$$

Proof. Before we investigate some of the cases mentioned in the above theorem, we should make the following observation. The inequality of the type (62) can be restated as following

$$(64) \quad H(x_1, x_2, x_3) \leq Cx_3,$$

where the function H is from (6) and $x_1 = \mathbb{E}F$, $x_2 = \mathbb{E}G$, $x_3 = \mathbb{E}|F|^p$. In order to derive the estimate (62) we have to find the sharp C in (64). Because of the property (31) we can assume that both of the values x_1, x_2 are nonnegative. So non-negativity of x_1, x_2 and the condition $|\mathbb{E}G| \leq \beta|\mathbb{E}F|$ can be reformulated as

$$(65) \quad -\frac{x_1 + x_2}{2} \leq \frac{x_2 - x_1}{2} \leq \left(\frac{\beta - 1}{\beta + 1} \right) \left(\frac{x_1 + x_2}{2} \right).$$

The condition (65) with (64) in terms of the function N and the variables y_1, y_2, y_3 means that we have to find the sharp C such that

$$N(y_1, y_2, y_3) \leq Cy_3 \quad \text{for} \quad -y_1 \leq y_2 \leq \left(\frac{\beta-1}{\beta+1}\right)y_1, \quad \mathbf{y} \in \Omega_2.$$

Because of (39), the above condition can be reformulated as following

$$(66) \quad \mathbf{B}(y_2, y_3) \leq Cy_3 \quad \text{for} \quad -1 \leq y_2 \leq \left(\frac{\beta-1}{\beta+1}\right), \quad y_3 \geq g(y_2),$$

where $\mathbf{B}(y_2, y_3) = N(1, y_2, y_3)$. So our aim is to find the sharp C , or in other words the value $\sup_{y_1, y_2} \mathbf{B}/y_3$ where the supremum is taken from the domain mentioned in (66). Note that, the quantity $\mathbf{B}(y_2, y_3)/y_3$ increase with respect to the variable y_2 . Indeed, $(\mathbf{B}(y_2, y_3)/y_3)'_{y_2} = t_1(s(y))/y_3$, where the function $t_1(s)$ is nonnegative on $[c_0, 1]$ (see the end of the proof of the concavity condition in the domain $\Omega(\Theta[s_0, y_p])$). Note that as we increase the value y_2 then the range of y_3 also increases. This means that the supremum of the expression \mathbf{B}/y_3 is attained on the subdomain, where $y_2 = (\beta-1)/(\beta+1)$. It is worth mentioning that because the quantity $(\beta-1)/(\beta+1) \in [-1, 1]$ increases as β increases and because of the observation made above, we see that the value C increase as the β' increases.

5.2. Case $y_p \leq s_0$. We are going to investigate the simple case (i). Remark 12 implies that $s_0 \leq y_p$, or in other words the foliation of vertical segments is the following $\Theta([y_p, 1], g)$ where the value $\theta(s)$ on $[y_p, 1]$ is equal to $\pi/2$. This means that $t_2(s)$ is constant on $[y_p, 1]$ (see Lemma 2), and it is equal to $f(y_p)/g(y_p) = (\tau^2 + \frac{1}{(p-1)^2})^{p/2}$ (see (50)).

If $\frac{\beta-1}{\beta+1} \geq y_p$, or equivalently $\beta \geq \frac{1}{p-1}$, then the function \mathbf{B} on the vertical segment with the endpoint $(\beta', g(\beta'))$ where $\frac{\beta-1}{\beta+1} = \beta' \in [y_p, 1]$ has the following expression (see (15))

$$\mathbf{B}(\beta', y_3) = f(\beta') + \frac{f(y_p)}{g(y_p)}(y_3 - g(\beta')), \quad y_3 \geq g(\beta').$$

Therefore

$$(67) \quad \frac{\mathbf{B}(\beta', y_3)}{y_3} = \frac{f(y_p)}{g(y_p)} + \frac{g(\beta')}{y_3} \left(\frac{f(\beta')}{g(\beta')} - \frac{f(y_p)}{g(y_p)} \right), \quad y_3 \geq g(\beta').$$

The expression $f(s)/g(s)$ is strictly increasing on $(-1, 1)$, therefore, the expression (67) attains its maximal value at point $y_3 = g(\beta')$, so we have

$$\begin{aligned} \frac{\mathbf{B}(y_2, y_3)}{y_3} &\leq \frac{\mathbf{B}(\beta', y_3)}{y_3} \leq \frac{\mathbf{B}(\beta', g(\beta'))}{g(\beta')} = \frac{f(\beta')}{g(\beta')} = \\ &(\tau^2 + \beta^2)^{p/2} \quad \text{for} \quad -1 \leq y_2 \leq \beta', \quad y_3 \geq g(y_2). \end{aligned}$$

If $\frac{\beta-1}{\beta+1} < y_p$, or equivalently $\beta < \frac{1}{p-1}$, then we can achieve such value for C which was achieved at moment $\beta = \frac{1}{p-1}$, and since the function $C = C(\beta')$ increases as β' increases, this value will be the best. Indeed, it suffice to look at the

foliation (see Figure 10). For any fixed y_2 we send y_3 to $+\infty$, and we obtain that

$$\lim_{y_3 \rightarrow \infty} \frac{\mathbf{B}(y_2, y_3)}{y_3} = \lim_{y_3 \rightarrow \infty} \frac{f(s) + t_1(s)(y_2 - s) + t_2(s)(y_3 - g(s))}{y_3} =$$

$$\lim_{y_3 \rightarrow \infty} t_2(s(y)) = t_2(y_p) = \left(\tau^2 + \frac{1}{(p-1)^2} \right)^{p/2}.$$

5.3. Case $y_p > s_0$. As it was already discussed, the condition in the case (ii) is equivalent to the inequality $s_0 > y_p$ (see Remark 12). This means that the foliation of vertical segments is $\Theta([s_0, 1], g)$ (see Figure 11). We know that $C(\beta')$ is increasing. We remind that we are going to maximize the function $\frac{\mathbf{B}(y_2, y_3)}{y_3}$ on the domain mentioned in (66). It was already mentioned that we can require $y_2 = \left(\frac{\beta-1}{\beta+1} \right) = \beta'$. For such fixed $y_2 = \beta' \in [-1, 1]$ we are going to investigate the monotonicity of the function $\frac{\mathbf{B}(\beta', y_3)}{y_3}$. We consider several cases. Let $\beta' \geq s_0$. We differentiate the function $\mathbf{B}(\beta', y_3)/y_3$ with respect to the variable y_3 , and we use the expression (15) for \mathbf{B} , so we obtain that

$$\frac{\partial}{\partial y_3} \left(\frac{\mathbf{B}(\beta', y_3)}{y_3} \right) = \frac{t_2(\beta')y_3 - \mathbf{B}(\beta', y_3)}{y_3^2} = \frac{-f(\beta') + t_2(\beta')g(\beta')}{y_3^2}.$$

Recall that $t_2(s) = t_2(s_0)$ for $s \in [s_0, 1]$, therefore, direct calculations imply

$$t_2(\beta') = \frac{f(s_0) - (s_0 - y_p)f'(s_0)}{g(s_0) - (s_0 - y_p)g'(s_0)} < \frac{f(s_0)}{g(s_0)} \leq \frac{f(\beta')}{g(\beta')}, \quad \beta' \geq s_0.$$

This implies that

$$C(\beta') = \sup_{y_3 \geq g(\beta')} \frac{\mathbf{B}(\beta', y_3)}{y_3} = \frac{\mathbf{B}(\beta', y_3)}{y_3} \Big|_{y_3=g(\beta')} = \frac{f(\beta')}{g(\beta')} = (\tau^2 + \beta^2)^{p/2}.$$

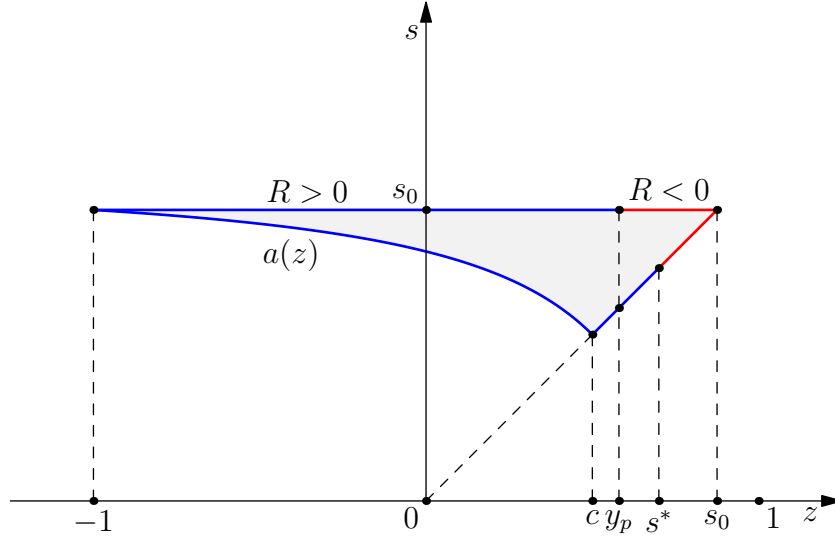
Now we consider the case $\beta' < s_0$.

For each point $y = (\beta', y_3)$ that belongs to the line $y_2 = \beta'$ there exists a segment $\ell(y) \in \Theta((c, s_0], g)$ with the endpoint $(s, g(s))$ where $s \in [\max\{\beta', \alpha(\beta')\}, s_0]$. If the point y belongs to the domain of linearity $\text{Ang}(s_0)$, then we can choose the value s_0 , and consider any segment with the endpoints y and $(s_0, g(s_0))$ which surely belongs to the domain of linearity. The reader can easily see that as we increase the value y_3 the value s increases as well. So,

$$\frac{\partial}{\partial y_3} \left(\frac{\mathbf{B}(\beta', y_3)}{y_3} \right) = \frac{t_2(s)y_3 - \mathbf{B}(\beta', y_3)}{y_3^2} = \frac{-f(s) - t_1(s)(\beta' - s) + t_2(s)g(s)}{y_3^2}.$$

Our aim is to investigate the sign of the expression $-f(s) - t_1(s)(\beta' - s) + t_2(s)g(s)$ as we variate the value $y_3 \in [g(\beta'), +\infty)$. Without loss of generality we can forget about the variable y_3 and, we can variate only the value s on an interval $[\max\{\alpha(\beta'), \beta'\}, s_0]$.

We consider the function $R(s, z) \stackrel{\text{def}}{=} -f(s) - t_1(s)(z - s) + t_2(s)g(s)$ with the following domain $-1 \leq z \leq s_0$ and $s \in [\max\{\alpha(z), z\}, s_0]$ (see Figure 12). As

FIGURE 12. Domain of $R(s, z)$

we already have seen $R(s_0, s_0) < 0$. Note that $R(s_0, -1) > 0$. Indeed, note that $R(s_0, -1) = t_2(s_0)g(-1) - f(-1)$. This equality follows from the fact that

$$f(s_0) - f(-1) = t_1(s_0)(s_0 + 1) + t_2(s_0)(g(s_0) - g(-1)),$$

which is consequence of Lemma 10. So, (52) and (28) imply

$$t_2(s_0) = \frac{f'(-1) - f'(s_0)}{g'(-1) - g'(s_0)} > \frac{f''(-1)}{g''(-1)} \geq \frac{f(-1)}{g(-1)}.$$

Note that the function $R(z, s_0)$ is linear with respect to z . So on the interval $[-1, s_0]$ it has a root which is $y_p = -1 + 2/p$. Indeed,

$$\frac{-f(s_0) + t_2(s_0)g(s_0) + t_1(s_0)s_0}{t_1(s_0)} = y_p.$$

The last equality follows from (52), (54) and (13). We need few more properties of the function $R(s, z)$. Note that for each fixed z , the function $R(s, z)$ is nonincreasing on $[\max\{\alpha(z), z\}, s_0]$. Indeed

$$(68) \quad R'_s(s, z) = -f'(s) - t'_1(s)(z - s) + t_1(s) + t'_2(s)g(s) + t_2(s)g'(s).$$

We take into account the condition (13), so the expression (68) simplifies into the following one

$$R'_s(s, z) = t'_2(s)g(s) + t'_1(s)(s - z).$$

We remind the reader equality (12) and the fact that $t'_2(s) \leq 0$. Therefore we have $R'_s(s, z) = y_3 t'_2(s)$ where $y_3 = y_3(s) > 0$. Thus we see that $R(s, \beta') \geq 0$ for $\beta' \leq y_p$.

So if the function $R(\cdot, z)$ at the right end on its domain $[\max\{\alpha(z), z\}, s_0]$ is positive, this will mean that the function \mathbf{B}/y_3 is increasing, hence, the constant

$C(\beta')$ will be equal to

$$C(\beta') = \lim_{y_3 \rightarrow \infty} \frac{\mathbf{B}(z, y_3)}{y_3} = t_2(s_0) = \frac{f'(-1) - f'(s_0)}{g'(-1) - g'(s_0)}$$

because of (52) and the structure of the foliation. By virtue of $u\left(\frac{1+s_0}{1-s_0}\right) = 0$ and (53), direct computations show that

$$(69) \quad \frac{f'(-1) - f'(s_0)}{g'(-1) - g'(s_0)} = \frac{\tau^p}{1 - \frac{2^{2-p}(1-s_0)^{p-1}}{(\tau^2+1)(p-1)(1-s_0)+2(2-p)}}.$$

So it follows that if $\beta' \leq y_p$ then (69) is the value of $C(\beta')$.

If the function $R(\cdot, z)$ on the left end of its domain is nonpositive, this will mean that the function \mathbf{B}/y_3 is decreasing, so the sharp constant will be the value of the function $\mathbf{B}(z, y_3)/y_3$ at the left end of its domain

$$(70) \quad C(\beta') = \frac{\mathbf{B}(z, y_3)}{y_3} \Big|_{y_3=g(z)} = \frac{f(z)}{g(z)} = (\tau^2 + \beta^2)^{p/2}.$$

We recall that c is the root of the cup and $c < y_p$ (see Remark 11). We will show that the function $R(z, s)$ is decreasing on the boundary $s = z$ for $s \in (y_p, s_0]$. Indeed, (13) implies

$$(R(s, s))'_s = -f'(s) + t'_2(s)g(s) + t_2(s)g'(s) = -t_1(s) + t'_2(s)g(s) < 0.$$

The last inequality follows from the fact that $t'_2(s) \leq 0$ and $t_1(s) > 0$ on $(c, 1]$. Surely $R(y_p, y_p) > R(s_0, y_p) = 0$, and we recall that $R(s_0, s_0) < 0$, so there exists unique $s^* \in [y_p, s_0]$ such that $R(s^*, s^*) = 0$. This is equivalent to (63). So it is clear that $R(z, z) \leq 0$ for $z \in [s^*, s_0]$. Therefore $C(\beta')$ has the value (70) for $\beta' \geq s^*$.

The only case remained is when $\beta' \in [y_p, s^*]$. We know that $R(z, z) \geq 0$ for $z \in [y_p, s^*]$ and $R(s_0, z) \leq 0$ for $z \in [y_p, s^*]$. The fact that for each fixed z the function $R(s, z)$ is decreasing implies the following: for each $z \in [y_p, s^*]$ there exists unique $s_1(z) \in [z, s_0]$ such that $R(z, s_1(z)) = 0$. Therefore, for $\beta' \in [y_p, s^*]$ we have

$$(71) \quad C(\beta') = \frac{\mathbf{B}(\beta', y_3(s_1(\beta')))}{y_3(s_1(\beta'))},$$

where the value $s_1(\beta')$ is the root of the equation $R(s_1(\beta'), \beta') = 0$. Recall that

$$(72) \quad \begin{aligned} R(s_1(\beta'), \beta') &= t_2(s_1)y_3(s_1) - \mathbf{B}(\beta', y_3(s_1)) \\ &= -f(s_1) - t_1(s_1)(\beta' - s_1) + t_2(s_1)g(s_1). \end{aligned}$$

So the expression (71) takes the form

$$C(\beta') = t_2(s_1) = \frac{f'(s_1) - f'(a(s_1))}{g'(s_1) - g'(a(s_1))}.$$

Finally, we remind the reader that

$$t_2(s) = \frac{f'(s) - f'(a(s))}{g'(s) - g'(a(s))},$$

$$t_1(s) = \frac{f'(a(s))g'(s) - f'(s)g'(a(s))}{g'(s) - g'(a(s))}.$$

for $s \in (c, s_0]$, and we finish the proof of the theorem. \square

6. EXTREMIZERS VIA FOLIATION

We set $\Psi(F, G) = \mathbb{E}(G^2 + \tau^2 F^2)^{2/p}$. Let N be a candidate that we have constructed in the Section 4 (see (55)). We define the candidate B for the Bellman function H (see (6)) in the following way

$$B(x_1, x_2, x_3) = N\left(\frac{x_1 + x_2}{2}, \frac{x_2 - x_1}{2}, x_3\right), \quad (x_1, x_2, x_3) \in \Omega.$$

We want to show that $B = H$. We already know that $B \geq H$ (see Lemma 3). So, it is remained to show that $B \leq H$. We are going to do this in the following way: for each point $\mathbf{x} \in \Omega$ and any $\varepsilon > 0$ we are going to find an admissible pair (F, G) such that

$$(73) \quad \Psi(F, G) > B(\mathbf{x}) - \varepsilon.$$

Up to the end of the current section we are going to work with the coordinates (y_1, y_2, y_3) (see (34)). It will be convenient for us to redefine notion of admissibility of the pair.

Definition 7. We say that a pair (F, G) is admissible for a point $(y_1, y_2, y_3) \in \Omega_1$, if G is a martingale transform of F and $\mathbb{E}(F, G, |F|^p) = (y_1 - y_2, y_1 + y_2, y_3)$.

So in this case condition (73) in terms of the function N takes the following form: for any point $\mathbf{y} \in \Omega_1$ and for any $\varepsilon > 0$ we are going to find an admissible pair (F, G) for the point \mathbf{y} such that

$$(74) \quad \Psi(F, G) > N(\mathbf{y}) - \varepsilon.$$

We formulate the following obvious observations.

Lemma 13. The following statements hold:

1. A pair (F, G) is admissible for a point $\mathbf{y} = (y_1, y_2, y_3)$ if and only if $(\tilde{F}, \tilde{G}) = (\pm F, \mp G)$ is admissible for a point $\tilde{\mathbf{y}} = (\mp y_2, \mp y_1, y_3)$, moreover $\Psi(\tilde{F}, \tilde{G}) = \Psi(F, G)$.
2. A pair (F, G) is an admissible for a point $\mathbf{y} = (y_1, y_2, y_3)$, if and only if $(\tilde{F}, \tilde{G}) = (\lambda F, \lambda G)$ where $\lambda \neq 0$ is an admissible pair for a point $\tilde{\mathbf{y}} = (\lambda y_1, \lambda y_2, |\lambda|^p y_3)$, moreover $\Psi(\tilde{F}, \tilde{G}) = |\lambda|^p \Psi(F, G)$.

Definition 8. Pair of functions (F, G) is called ε -extremizer for a point $\mathbf{y} \in \Omega_1$ if (F, G) is an admissible pair for the point \mathbf{y} and $\Psi(F, G) > N(\mathbf{y}) - \varepsilon$.

Lemma 13, homogeneity and symmetry of N imply that we only need to check (74) for the points $\mathbf{y} \in \Omega_1$ where $y_1 = 1$ ($y_2, y_3 \in \Omega_3$). In other words, we show that $\Psi(F, G) > \mathbf{B}(y_2, y_3) - \varepsilon$ for some admissible (F, G) of the point $(1, y_2, y_3)$ where $(y_2, y_3) \in \Omega_3$. Further, instead of saying that (F, G) is an admissible pair (or ε -extremizer) for the point $(1, y_2, y_3)$ we just say that it is admissible pair (or ε -extremizer) for the point (y_2, y_3) . So we only have to construct ε -extremizers in the domain Ω_3 .

It is worth mentioning that we construct ε -extremizers (F, G) such that G will be a martingale transform of F with respect to some filtration other than dyadic. Detailed explanation how to pass from one filtration to another one reader can find in [10].

We need few more observations. For $\alpha \in (0, 1)$ we define α concatenation of the pairs (F, G) and (\tilde{F}, \tilde{G}) in the following way

$$(F \bullet \tilde{F}, G \bullet \tilde{G})_\alpha(x) = \begin{cases} (F, G)(x/\alpha) & x \in [0, \alpha], \\ (\tilde{F}, \tilde{G})((x - \alpha)/(1 - \alpha)) & x \in [\alpha, 1]. \end{cases}$$

Clearly $\Psi((F \bullet \tilde{F}, G \bullet \tilde{G})_\alpha(x)) = \alpha\Psi(F, G) + (1 - \alpha)\Psi(\tilde{F}, \tilde{G})$.

Definition 9. Any domain of the type $\Omega_1 \cap \{y_1 = A\}$ where A is some real number is said to be positive domain. Any domain of the type $\Omega_1 \cap \{y_2 = B\}$ where B is some real number is said to be negative domain.

The following lemma is obvious.

Lemma 14. If (F, G) is an admissible pair for a point \mathbf{y} and (\tilde{F}, \tilde{G}) is an admissible pair for a point $\tilde{\mathbf{y}}$ such that either of the following is true: $\mathbf{y}, \tilde{\mathbf{y}}$ belong to a positive domain, or $\mathbf{y}, \tilde{\mathbf{y}}$ belong to a negative domain, then $(F \bullet \tilde{F}, G \bullet \tilde{G})_\alpha$ is an admissible pair for the point $\alpha\mathbf{y} + (1 - \alpha)\tilde{\mathbf{y}}$.

Let (F, G) be an admissible pair for a point \mathbf{y} , and let (\tilde{F}, \tilde{G}) be an admissible pair for a point $\tilde{\mathbf{y}}$. Let $\hat{\mathbf{y}}$ be a point which belongs to the chord joining the points \mathbf{y} and $\tilde{\mathbf{y}}$.

Remark 14. It is clear that if (F^+, G^+) is an admissible for a point (y_2^+, y_3^+) and (F^-, G^-) is an admissible for a point (y_2^-, y_3^-) then α concatenation of these pairs is admissible for the point $(y_2, y_3) = \alpha \cdot (y_2^+, y_3^+) + (1 - \alpha) \cdot (y_2^-, y_3^-)$.

Now we are ready to construct ε -extremizers in Ω_3 . The main idea is that these functions Ψ and \mathbf{B} are very similar: they have almost the same properties, and the crucial role plays foliation.

6.1. Case $s_0 \leq y_p$. We want to find ε -extremizers for the points in Ω_3 .

Extremizers in the domain $\Omega(\Theta_{\text{cup}}((c, s_0], g))$.

Pick any $y = (y_2, y_3) \in \Omega(\Theta_{\text{cup}}((c, s_0], g))$. Then there exists a segment $\ell(y) \in \Theta_{\text{cup}}((c, s_0], g)$. Let $y^+ = (s, g(s))$ and $y^- = (a(s), g(a(s)))$ be the endpoints of $\ell(y)$ in Ω_3 . We know ε -extremizers at these points y^+, y^- . Indeed, we can take the following ε -extremizers $(F^+, G^+) = (1 - s, 1 + s)$ and $(F^-, G^-) = (1 - a(s), 1 +$

$a(s)$ (i.e. constant functions). Consider an α concatenation $(F^+ \bullet F^-, G^+ \bullet G^-)_\alpha$, where α is chosen so that $y = \alpha y^+ + (1 - \alpha)y^-$. We have

$$\begin{aligned} \Psi[(F^+ \bullet F^-, G^+ \bullet G^-)_\alpha] &= \alpha \Psi(F^+, G^+) + (1 - \alpha) \Psi(F^-, G^-) > \\ \alpha \mathbf{B}(y^+) + (1 - \alpha) \mathbf{B}(y^-) - \varepsilon &= \mathbf{B}(y) - \varepsilon. \end{aligned}$$

Last equality follows from linearity of \mathbf{B} on $\ell(y)$.

Extremizers on vertical line $(-1, y_3), y_3 \geq h(s_0)$.

Now we are going to find ε -extremizers for the points $(-1, y_3)$ where $y_3 \geq h(s_0)$. We use an idea mentioned in [5] (see proof of Lemma 3). We define the functions (F, G) recursively:

$$G(t) = \begin{cases} -w & 0 \leq t < \varepsilon; \\ \gamma \cdot g\left(\frac{t-\varepsilon}{1-2\varepsilon}\right) & \varepsilon \leq t \leq 1 - \varepsilon; \\ w & 1 - \varepsilon < t \leq 1; \end{cases}$$

$$F(t) = \begin{cases} d_- & 0 \leq t < \varepsilon; \\ \gamma \cdot f\left(\frac{t-\varepsilon}{1-2\varepsilon}\right) & \varepsilon \leq t \leq 1 - \varepsilon; \\ d_+ & 1 - \varepsilon < t \leq 1; \end{cases}$$

where the nonnegative constants w, d_-, d_+, γ will be obtained from the requirement $\mathbb{E}(F, G, |F|^p) = (2, 0, y_3)$ and the fact that G is a martingale transform of F . Surely $\langle G \rangle_{[0,1]} = 0$. Condition $\langle F \rangle_{[0,1]} = 2$ means that

$$(75) \quad (d_- + d_+) \varepsilon + 2\gamma(1 - 2\varepsilon) = 2.$$

Condition $\langle |F|^p \rangle_{[0,1]} = y_3$ implies that

$$(76) \quad y_3 = \frac{\varepsilon(d_+^p + d_-^p)}{1 - (1 - 2\varepsilon)\gamma^p}.$$

Now we use the condition $|F_0 - F_1| = |G_0 - G_1|$. At first time we split the interval $[0, 1]$ at point ε with requirement $F_0 - F_1 = G_0 - G_1$. We obtain that $w = 2 - d_-$. On the second time we split at point $1 - \varepsilon$ with requirement $F_1 - F_2 = G_2 - G_1$ then we obtain $w = 2\gamma - d_+$. From these two conditions we obtain $d_- + d_+ = 2(1 + \gamma) - 2w$, and by substituting in (75) we find the γ

$$\gamma = 1 + \frac{\varepsilon w}{1 - \varepsilon}.$$

Now we investigate what happens as ε tends to zero. Our aim will be to focus on the limit value $\lim_{\varepsilon \rightarrow 0} w = w_0$. We have $1 - (1 - 2\varepsilon)\gamma^p \approx \varepsilon(2 - wp)$. So (76) becomes

$$(77) \quad y_3 = \frac{\varepsilon(d_+^p + d_-^p)}{1 - (1 - 2\varepsilon)\gamma^p} \rightarrow \frac{2(2 - w_0)^p}{2 - w_0 p}.$$

Note that for $w_0 = 1 + s$ equation (77) is the same as (48). By direct calculations we see that as $\varepsilon \rightarrow 0$ we have

$$\langle (G^2 + \tau^2 F^2)^{p/2} \rangle_{[0,1]} = \frac{\varepsilon[(w^2 + \tau^2 d_-^2)^{p/2} + (w^2 + \tau^2 d_+^2)^{p/2}]}{1 - (1 - 2\varepsilon)\gamma^p} \rightarrow \frac{2f(w_0 - 1)}{2 - w_0 p}.$$

Now we are going to calculate the value $\mathbf{B}(-1, h(s))$ where $h(s) = y_3$. From (46) we have

$$\mathbf{B}(-1, h(s)) = h(s)t_2(s) - \frac{2}{p}t_1(s).$$

By using (13) we express t_1 via t_2 , also because of (48) and (51) we have

$$\begin{aligned} \mathbf{B}(-1, h(s)) &= h(s)t_2(s) - \frac{2}{p}t_1(s) = h(s)t_2 - \frac{2}{p}(f' - t_2g') = \\ t_2(h(s) + \frac{2}{p}g') - f' \frac{2}{p} &= \frac{f(s) - (s - y_p)f'(s)}{g(s) - (s - y_p)g'(s)} \left(\frac{2g}{p(y_p - s)} + \frac{2}{p}g' \right) - \\ - f' \frac{2}{p} &= \frac{2}{p} \left[\frac{f(s)}{y_p - s} \right] = \frac{2(2 - w_0)^p}{2 - w_0 p}. \end{aligned}$$

Thus we obtain the desired result

$$\langle (G^2 + \tau^2 F^2)^{p/2} \rangle_{[0,1]} \rightarrow \mathbf{B}(-1, y_3) \quad \text{as } \varepsilon \rightarrow 0.$$

Extremizers in the domain $\Omega(\Theta([s_0, y_p], g))$.

Pick any point $y = (y_2, y_3) \in \Omega(\Theta([s_0, y_p], g))$. Then there exists a segment $\ell(y) \in \Theta([s_0, y_p], g)$. Let y^+ and y^- be the endpoints of this segment such that $y^+ = (-1, \tilde{y}_3)$ for some $\tilde{y}_3 \geq h(s_0)$ and $y^- = (\tilde{s}, g(\tilde{s}))$ for some $\tilde{s} \in [y_p, s_0]$. We remind the reader that we know ε -extremizers for the points $(s, g(s))$ where $s \in [s_0, 1]$, and we know ε -extremizers on vertical line $(-1, y_3)$ where $y_3 \geq h(s_0)$. Therefore, as in the case of a cup, taking corresponding concatenation of these ε -extremizers and using the fact that \mathbf{B} is linear on $\ell(y)$, we find ε -extremizer at point y .

Extremizers in the domain $\text{Ang}(s_0)$.

Pick any $y = (y_1, y_2) \in \text{Ang}(s_0)$. There exist the points $y^+ \in \ell^+$, $y^- \in \ell^-$, where $\ell^+ = \ell^+(s_0, g(s_0)) \in \Theta([s_0, y_p], g)$ and $\ell^- = \ell^-(s_0, g(s_0)) \in \Theta([c, s_0], g)$, such that $y = \alpha y^+ + (1 - \alpha)y^-$ for some $\alpha \in [0, 1]$. We know ε -extremizers at points y^+ and y^- . Then by taking α concatenation of these extremizers and using the linearity of \mathbf{B} on $\text{Ang}(s_0)$ we can obtain ε -extremizer at point y .

Extremizers in the domain $\Omega(\Theta([y_p, 1], g))$.

Finally, we consider domain of vertical segments $\Omega(\Theta([y_p, 1], g))$. Pick any point $y = (y_2, y_3) \in \Omega(\Theta([y_p, 1], g))$. Take an arbitrary point $y^+ = (-1, y_3^+)$ where y_3^+ is sufficiently large such that $y = \alpha y^+ + (1 - \alpha)y^-$ for some $\alpha \in (0, 1)$ and some $y^- = (y_2^-, y_3^-)$ such that $(1, y_2^-, y_3^-) \in \partial\Omega_1$. Surely, y^+, y^- belong to a positive domain. Condition $(1, y_2^-, y_3^-) \in \partial\Omega_1$ implies that we know ε -extremizer (F^-, G^-) at point y^- (these are constant functions). We also know ε -extremizer at point y^+ . Let $(F^+ \bullet F^-, G^+ \bullet G^-)_\alpha$ be α concatenation of these extremizers. Then

$$\Psi[(F^+ \bullet F^-, G^+ \bullet G^-)_\alpha] > \alpha \mathbf{B}(y^+) + (1 - \alpha) \mathbf{B}(y^-) - \varepsilon.$$

Note that the condition $y = \alpha y^+ + (1 - \alpha)y^-$ implies that

$$\alpha = \frac{y_3 - \frac{y_2}{y_2^-} y_3^-}{y_3^+ + \frac{y_3^-}{y_2^-}}.$$

Recall that $\mathbf{B}(y_2, g(y_2)) = f(y_2)$ and $\mathbf{B}(y^+) = f(s) + t_1(s)(-1 - s) + t_2(s)(y_3^+ - g(s))$, where $s \in [s_0, y_p]$ is such that a segment $\ell(s, g(s)) \in \Theta([s_0, y_p], g)$ has an endpoint y^+ .

Note that as $y_3^+ \rightarrow \infty$ all terms remain bounded, moreover, $\alpha \rightarrow 0$, $y^- \rightarrow (y_2, g(y_2))$ and $s \rightarrow y_p$. This means that

$$\begin{aligned} \lim_{y_3^+ \rightarrow \infty} \alpha \mathbf{B}(y^+) + (1 - \alpha) \mathbf{B}(y^-) - \varepsilon &= \\ \lim_{y_3^+ \rightarrow \infty} t_2(s) y_3^+ \left(\frac{y_3 - \frac{y_2}{y_2^-} y_3^-}{y_3^+ + \frac{y_3^-}{y_2^-}} \right) + f(y_2) - \varepsilon &= \\ t_2(y_p)(y_3 - g(y_2)) + f(y_2) - \varepsilon. \end{aligned}$$

We recall that $t_2(s) = t_2(y_p)$ for $s \in [y_p, 1]$. Then

$$\mathbf{B}(y) = f(y_2) + t_2(s(y))(y_3 - g(y_2)) = f(y_2) + t_2(y_p)(y_3 - g(y_2)).$$

Thus, if we choose y_3^+ sufficiently large then we can obtain 2ε -extremizer for the point y .

6.2. Case $s_0 > y_p$. In this case we have $s_0 \geq y_p$ (see Figure 11). This case is a little bit complicated than previous one. Construction of ε -extremizers (F, G) will be similar to the one presented in [11].

We need few more definitions. Let (F, G) be an arbitrary pair of functions.

Definition 10. We say that for a pair (F, G) we put a constant $(y_2, g(y_2)) \in \Omega_3$ on an interval $J \subseteq [0, 1]$ if instead of pair (F, G) we consider a new pair (\tilde{F}, \tilde{G}) such that

$$(\tilde{F}, \tilde{G})(x) = \begin{cases} (F, G)(x) & x \in [0, 1] \setminus J \\ (1 - y_2, 1 + y_2) & x \in J. \end{cases}$$

It is worth mentioning that sometimes the new pair (\tilde{F}, \tilde{G}) we denote by the same symbol (F, G) .

Definition 11. We say that the pairs (F_α, G_α) , $(F_{1-\alpha}, G_{1-\alpha})$ are obtained from the pair (F, G) by splitting at point $\alpha \in (0, 1)$ if

$$\begin{aligned} (F_\alpha, G_\alpha) &= (F, G)(x \cdot \alpha) \quad x \in [0, 1]; \\ (F_{1-\alpha}, G_{1-\alpha}) &= (F, G)((x + \alpha) \cdot (1 - \alpha)) \quad x \in [0, 1]; \end{aligned}$$

It is clear that $\Psi(F, G) = \alpha \Psi(F_\alpha, G_\alpha) + (1 - \alpha) \Psi(F_{1-\alpha}, G_{1-\alpha})$. Also note that if (F_α, G_α) , $(F_{1-\alpha}, G_{1-\alpha})$ are obtained from the pair (F, G) by splitting at point $\alpha \in (0, 1)$, then (F, G) is α concatenation of the pairs (F_α, G_α) , $(F_{1-\alpha}, G_{1-\alpha})$. Thus, such operation as splitting and concatenation are opposite operations.

Instead of explicitly presenting an admissible pair (F, G) and showing that it is ε -extremizer, we present an algorithm which constructs the admissible pair, and we show that the result is ε -extremizer.

By the same explanations as in the case $s_0 \leq y_p$, it is enough to construct ε -extremizer (F, G) on the vertical line $y_2 = -1$ of the domain Ω_3 . Moreover, linearity of \mathbf{B} implies that for any $A > 0$, it is enough to construct ε -extremizers for the points $(-1, y_3)$, where $y_3 \geq A$. Pick any point $(-1, y_3)$ where $y_3 = y_3^{(0)} > g(-1)$. Linearity of \mathbf{B} on $\text{Ang}(s_0)$ and direct calculations (see (15), (52)) show that

$$(78) \quad \begin{aligned} \mathbf{B}(-1, y_3) &= f(-1) + t_3(s_0)(y_3 - g(-1)) = \\ &= f(-1) + (y_3 - g(-1)) \frac{f'(-1) - f'(s_0)}{g'(-1) - g'(s_0)}. \end{aligned}$$

We describe the first iteration. Let (F, G) be an admissible pair for the point $(-1, y_3)$ explicit expression of which will be described during the algorithm. For a pair (F, G) we put a constant $(s_0, g(s_0))$ on an interval $[0, \varepsilon]$ where the value $\varepsilon \in (0, 1)$ will be given later. Thus we obtain a new pair (F, G) which we denote by the same symbol. We want (F, G) to be an admissible pair for the point $(-1, y_3)$. Let the pairs $(F_\varepsilon, G_\varepsilon)$, $(F_{1-\varepsilon}, G_{1-\varepsilon})$ be obtained from the pair (F, G) by splitting at point ε . It is clear that $(F_\varepsilon, G_\varepsilon)$ is an admissible pair for the point $(s_0, g(s_0))$. We want $(F_{1-\varepsilon}, G_{1-\varepsilon})$ to be an admissible pair for the point $P = (\tilde{y}_2, \tilde{y}_3)$ so that

$$(79) \quad (-1, y_3) = \varepsilon(s_0, g(s_0)) + (1 - \varepsilon)P.$$

Therefore we require

$$(80) \quad P = \left(\frac{-1 - \varepsilon s_0}{1 - \varepsilon}, \frac{y_3 - \varepsilon g(s_0)}{1 - \varepsilon} \right).$$

So we make the following simple observation: if $(F_{1-\varepsilon}, G_{1-\varepsilon})$ were an admissible pair for the point P , then (F, G) (which is ε concatenation of the pairs $(1 - s_0, 1 + s_0)$ and $(F_{1-\varepsilon}, G_{1-\varepsilon})$) would be an admissible pair for the point $(-1, y_3)$. Explanation of this observation is simple: note that these pairs $(F_{1-\varepsilon}, G_{1-\varepsilon})$ and $(1 - s_0, 1 + s_0)$ are admissible pairs for the points P and $(s_0, g(s_0))$ which belong to a positive domain (see (79)), therefore, the rest immediately follows from Lemma 14. So we want to construct the admissible pair $(F_{1-\varepsilon}, G_{1-\varepsilon})$ for the point (80).

We recall Lemma 13 which implies that the pair $(F_{1-\varepsilon}, G_{1-\varepsilon})$ is admissible for the point $\left(1, \frac{-1 - \varepsilon s_0}{1 - \varepsilon}, \frac{y_3 - \varepsilon g(s_0)}{1 - \varepsilon}\right)$ if and only if a pair (\tilde{F}, \tilde{G}) where

$$(F_{1-\varepsilon}, -G_{1-\varepsilon}) = \frac{1 + \varepsilon s_0}{1 - \varepsilon} (\tilde{F}, \tilde{G})$$

is admissible for a point $W = \left(1, \frac{\varepsilon - 1}{1 + \varepsilon s_0}, \frac{(y_3 - \varepsilon g(s_0))}{(1 + \varepsilon s_0)^p} \cdot (1 - \varepsilon)^{p-1}\right)$. So, if we find the admissible pair (\tilde{F}, \tilde{G}) then we automatically find the admissible pair (F, G) .

Choose ε small enough so that $\left(\frac{\varepsilon - 1}{1 + \varepsilon s_0}, \frac{(y_3 - \varepsilon g(s_0))}{(1 + \varepsilon s_0)^p} \cdot (1 - \varepsilon)^{p-1}\right) \in \Omega_3$ and

$$\left(\frac{\varepsilon - 1}{1 + \varepsilon s_0}, \frac{(y_3 - \varepsilon g(s_0))}{(1 + \varepsilon s_0)^p} \cdot (1 - \varepsilon)^{p-1}\right) = \delta(s_0, g(s_0)) + (1 - \delta)(-1, y_3^{(1)})$$

for some $\delta \in (0, 1)$ and $y_3^{(1)} \geq g(-1)$. Then

$$\begin{aligned}
 \delta &= \frac{\varepsilon}{1 + \varepsilon s_0} = \varepsilon + O(\varepsilon^2) \\
 y_3^{(1)} &= \frac{\frac{(y_3 - \varepsilon g(s_0))}{(1 + \varepsilon s_0)^p} \cdot (1 - \varepsilon)^{p-1} - \frac{\varepsilon}{1 + \varepsilon s_0} g(s_0)}{1 - \frac{\varepsilon}{1 + \varepsilon s_0}} = \\
 (81) \quad &= y_3(1 - \varepsilon(p + ps_0 - 2)) - 2\varepsilon g(s_0) + O(\varepsilon^2).
 \end{aligned}$$

For the pair (\tilde{F}, \tilde{G}) we put a constant $(s_0, g(s_0))$ on the interval $[0, \delta]$. We split the new pair (\tilde{F}, \tilde{G}) at point δ so we get the pairs $(\tilde{F}_\delta, \tilde{G}_\delta)$ and $(\tilde{F}_{1-\delta}, \tilde{G}_{1-\delta})$. We make similar observation as above. It is clear that if we know the admissible pair $(\tilde{F}_{1-\delta}, \tilde{G}_{1-\delta})$ for the point $(-1, y_3^{(1)})$ then we can obtain admissible pair (\tilde{F}, \tilde{G}) for the point $\left(\frac{\varepsilon-1}{1+\varepsilon s_0}, \frac{(y_3 - \varepsilon g(s_0))}{(1+\varepsilon s_0)^p} \cdot (1 - \varepsilon)^{p-1}\right)$. Surely (\tilde{F}, \tilde{G}) is δ concatenation of the pairs $(1 - s_0, 1 + s_0)$ and $(\tilde{F}_{1-\delta}, \tilde{G}_{1-\delta})$.

We summarize the first iteration. We took $\varepsilon \in (0, 1)$, and we started from the pair $(F^{(0)}, G^{(0)}) = (F, G)$, and after one iteration we came to the pair $(F^{(1)}, G^{(1)}) = (\tilde{F}_{1-\delta}, \tilde{G}_{1-\delta})$. We showed that if $(F^{(1)}, G^{(1)})$ is an admissible pair for the point $(1, y_3^{(1)})$ then the pair $(F^{(0)}, G^{(0)})$ can be obtained from the pair $(F^{(1)}, G^{(1)})$, moreover, it is admissible for the point $(1, y_3^{(0)})$.

Continuing these iterations, we obtain sequence of numbers $\{y_3^{(j)}\}_{j=0}^N$ and the sequence of pairs $\{(F^{(j)}, G^{(j)})\}_{j=0}^N$. Let N be such that $y_3^{(N)} \geq g(-1)$. It is clear that if $(F^{(N)}, G^{(N)})$ is an admissible pair for the point $(-1, y_3^{(N)})$ then the pairs $\{(F^{(j)}, G^{(j)})\}_{j=0}^{N-1}$ can be determined uniquely, and, moreover, $(F^{(j)}, G^{(j)})$ is an admissible pair for the point $(-1, y_3^{(j)})$ for all $j = 0, \dots, N-1$.

Note that we can choose sufficiently small $\varepsilon \in (0, 1)$, and we can find $N = N(\varepsilon)$ such that $y_3^{(N)} = g(-1)$ (see (81), and recall that $s_0 > y_p$). In this case admissible pair $(F^{(N)}, G^{(N)})$ for the point $(-1, y_3^{(N)}) = (-1, g(-1))$ is a constant function, namely, $(F^{(N)}, G^{(N)}) = (2, 0)$. Now we try to find N in terms of ε , and we try to find the value of $\Psi(F^{(0)}, G^{(0)})$.

Condition (81) implies that $y_3^{(1)} = y_3^{(0)}(1 - \varepsilon(p + ps_0 - 2)) - 2\varepsilon g(s_0) + O(\varepsilon^2)$. We denote $\delta_0 = p + ps_0 - 2 > 0$. Therefore, after N -th iteration we obtain

$$y_3^{(N)} = (1 - \varepsilon \delta_0)^N \left(y_3^{(0)} + \frac{2g(s_0)}{\delta_0} \right) - \frac{2g(s_0)}{\delta_0} + O(\varepsilon).$$

The requirement $y_3^{(N)} = g(-1)$ implies that

$$(1 - \varepsilon \delta_0)^{-N} = \frac{y_3^{(0)} + \frac{2g(s_0)}{\delta_0}}{g(-1) + \frac{2g(s_0)}{\delta_0}} + O(\varepsilon).$$

This implies that $\limsup_{\varepsilon \rightarrow 0} \varepsilon \cdot N = \limsup_{\varepsilon \rightarrow 0} \varepsilon \cdot N(\varepsilon) < \infty$. Therefore, we get

$$(82) \quad e^{\varepsilon \delta_0 N} = \frac{y_3^{(0)} + \frac{2g(s_0)}{\delta_0}}{g(-1) + \frac{2g(s_0)}{\delta_0}} + O(\varepsilon).$$

Also note that

$$\begin{aligned} \Psi(F^{(0)}, G^{(0)}) &= \Psi(F, G) = \varepsilon \Psi(F_\varepsilon, G_\varepsilon) + (1 - \varepsilon) \Psi(F_{1-\varepsilon}, G_{1-\varepsilon}) = \\ &= \varepsilon f(s_0) + (1 - \varepsilon) \Psi(F_{1-\varepsilon}, G_{1-\varepsilon}) = \varepsilon f(s_0) + (1 - \varepsilon) \left(\frac{1 + \varepsilon s_0}{1 - \varepsilon} \right)^p \Psi(\tilde{F}, \tilde{G}) \\ &= \varepsilon f(s_0) + (1 - \varepsilon)(1 - \varepsilon) \left(\frac{1 + \varepsilon s_0}{1 - \varepsilon} \right)^p [\delta f(s_0) + (1 - \delta) \Psi(\tilde{F}_{1-\delta}, \tilde{G}_{1-\delta})] \\ &= 2\varepsilon f(s_0) + (1 + \varepsilon \delta_0) \Psi(F^{(1)}, G^{(1)}) + O(\varepsilon^2). \end{aligned}$$

Therefore, after N -th iteration (and using the fact that $\Psi(F^{(N)}, G^{(N)}) = f(-1)$) we obtain

$$(83) \quad \begin{aligned} \Psi(F^{(0)}, G^{(0)}) &= (1 + \varepsilon \delta_0)^N \left(f(-1) + \frac{2f(s_0)}{\delta_0} \right) - \frac{2f(s_0)}{\delta_0} + O(\varepsilon) = \\ e^{\varepsilon \delta_0 N} \left(f(-1) + \frac{2f(s_0)}{\delta_0} \right) &- \frac{2f(s_0)}{\delta_0} + O(\varepsilon). \end{aligned}$$

The last equality follows from the fact that $\limsup_{\varepsilon \rightarrow 0} \varepsilon \cdot N(\varepsilon) < \infty$.

Therefore (82) and (83) imply that

$$\begin{aligned} \Psi(F^{(0)}, G^{(0)}) &= \left(\frac{y_3^{(0)} + \frac{2g(s_0)}{\delta_0}}{g(-1) + \frac{2g(s_0)}{\delta_0}} \right) \left(f(-1) + \frac{2f(s_0)}{\delta_0} \right) - \frac{2f(s_0)}{\delta_0} + O(\varepsilon) = \\ f(-1) + (y_3^{(0)} - g(-1)) &\left(\frac{f(-1) + \frac{2f(s_0)}{\delta_0}}{g(-1) + \frac{2g(s_0)}{\delta_0}} \right) + O(\varepsilon). \end{aligned}$$

Now we recall (78). So if we show that

$$(84) \quad \frac{f(-1) + \frac{2f(s_0)}{\delta_0}}{g(-1) + \frac{2g(s_0)}{\delta_0}} = \frac{f'(-1) - f'(s_0)}{g'(-1) - g'(s_0)}$$

then (84) will imply that $\Psi(F^{(0)}, G^{(0)}) = \mathbf{B}(-1, y_3^{(0)}) + O(\varepsilon)$. So choosing ε sufficiently small we can obtain the extremizer $(F^{(0)}, G^{(0)})$ for the point $(-1, y_3)$. Therefore, we only need to prove equality (84). It will be convenient to make the following notations. Set $f_- = f(-1)$, $f'_- = f'(-1)$, $f = f(s_0)$, $f' = f'(s_0)$, $g_- = g(-1)$, $g'_- = g'(-1)$, $g = g(s_0)$ and $g' = g'(s_0)$. Then the equality (84) turns into the following one

$$(85) \quad \frac{\delta_0}{2} = \frac{f g'_- - f g' - f'_- g + f' g}{g' f_- - f' g_-}.$$

This simplifies into the following one

$$s_0 - y_p = \frac{2}{p} \cdot \frac{fg'_- - fg' - f'_-g + f'g}{g'f_- - f'g_-} = \frac{fg'_- - fg' - f'_-g + f'g}{-g'f'_- + f'g'_-}$$

which is true by (54).

ACKNOWLEDGEMENTS

I would like to express my deep gratitude to Professor A. Volberg, Professor V. I. Vasyunin and Professor S. V. Kislyakov, my research supervisors, for their patient guidance, enthusiastic encouragement and useful critiques of this work. I would also like to thank A. Reznikov, for his assistance in finding ε -extremizers for the Bellman function. I would also like to extend my thanks to my colleagues and close friends P. Zatitskiy, N. Osipov and D. Stolyarov for working together in Saint-Petersburg, and developing a theory for minimal concave functions.

Finally, I wish to thank my parents for their support and encouragement throughout my study.

REFERENCES

- [1] F. Nazarov, S. Treil, and A. Volberg, *Bellman function in stochastic control and harmonic analysis*, (Bordeaux, 2000) Oper. Theory Adv. Appl., vol. 129, Birkhauser, Basel, 2001, pp. 393-423. MR 1882704 (2003b:49024)
- [2] A. Volberg, *Bellman function technique in Harmonic Analysis*. Lectures of INRIA Summer School in Antibes, June 2011. <http://arxiv.org/abs/1106.3899>
- [3] B. Simon, *Convexity: An Analytic Viewpoint*. Cambridge University Press (2011). p. 287. ISBN 1-10iOS70-0731-3.
- [4] N. Boros, P. Janakiraman, A. Volberg, *Perturbation of Burkholder's martingale transform and Monge-Ampère equation* <http://arxiv.org/abs/1102.3905>
- [5] V. Vasyunin, A. Volberg, Burkholder's function via Monge-Ampère equation, <http://arxiv.org/abs/1006.2633>
- [6] P. Ivanishvili, N. N. Osipov, D. M. Stolyarov, V. I. Vasyunin, P. B. Zatitskiy, *Bellman function for extremal problems on BMO* <http://arxiv.org/abs/1205.7018>
- [7] A. V. Pogorelov, *Differential Geometry*, "Noordhoff" 1959.
- [8] D. L. Burkholder, *Boundary value problem and sharp inequalities for martingale transforms*, Ann. Probab. 12 (1984), 647-702.
- [9] K. P. Choi, *A sharp inequality for martingale transforms and the unconditional basis constant of a monotone basis in $L_p(0,1)$* , Trans. Amer. Math. Soc. 330 (1992), no. 2, 509-529. MR 1034661 (92f:60073), <http://dx.doi.org/10.1090/S0002-9947-1992-1034661-3>
- [10] L. Slavin, V. Vasyunin, *Sharp results in the integral-form John-Nirenberg inequality*, Trans. Amer. Math. Soc. 2007; Preprint <http://arxiv.org/abs/0709.4332>
- [11] A. Reznikov, V. Vasyunin, V. Volberg *Extremizers and Bellman function for martingale weak type inequality*, 2013; Preprint <http://arxiv.org/abs/1311.2133>

DEPARTMENT OF MATHEMATICS, MICHIGAN STATE UNIVERSITY, EAST LANSING, MI 48824, USA

E-mail address: ivanisvi@math.msu.edu

URL: <http://math.msu.edu/~ivanisvi>